So far we've treated the CMB as unpolarized. This is not quite correct, and it matters for two reasons. One is that polarization affects the Thomson scattering cross section so polarization anisotropies feed back into the temperature anisotropies. This is a percent-level effect but important for precision cosmology. Much more important is that the polarization adds a new window on the processes of recombination and reionization, the nature of the primordial perturbations, and searches for tensor perturbations.

1 Basic theory

Description. Up until now we've assumed that the photons have a scalar phase space density $f(x^i, p, \hat{p}^i, \eta)$. In reality there are two photon polarizations, vertical ($\hat{\theta}$) and horizontal ($\hat{\phi}$), and the phase space density may be different in each one. Moreover the two polarizations may be correlated if the photons are polarized in a diagonal direction or have circular polarization. Generally we write the phase space density as a 2×2 Hermitian density matrix:

$$f = \begin{pmatrix} f_{\hat{\theta}\hat{\theta}} & f_{\hat{\theta}\hat{\phi}} \\ f_{\hat{\phi}\hat{\theta}} & f_{\hat{\phi}\hat{\phi}} \end{pmatrix} = \begin{pmatrix} f_I + f_Q & f_U + i f_V \\ f_U - i f_V & f_I - f_Q \end{pmatrix}. \tag{1}$$

The phase space density viewed through a linear polarizing filter at position angle ψ is

$$f(x^i, p, \hat{p}^i, \eta; \psi) = f_I + f_Q \cos 2\psi + f_U \sin 2\psi. \tag{2}$$

The temperature perturbation that we have been studying has an analogue for polarization. We define the temperature polarization Θ as:

$$f_I = \left[\exp \frac{p}{T_{\gamma 0}(1+\Theta)} - 1 \right]^{-1}, \tag{3}$$

or

$$\Theta(x^i, p, \hat{p}^i, \eta) = \frac{f_I - f^{(0)}}{-p\, \partial f^{(0)}/\partial p}. \tag{4}$$

In the unperturbed Universe there is no polarization. Therefore we may define

$$Q(x^i, p, \hat{p}^i, \eta) = \frac{f_Q}{-p\, \partial f^{(0)}/\partial p}, \tag{5}$$

and similarly for U and V. Our job is thus to follow Q, U, and V in addition to Θ. As usual we will work in terms of the Fourier modes, i.e. use k_i instead of x^i, and arrange to put the 3rd coordinate axis in the direction of $\mathbf{k}$.

The polarization will be created by Thomson scattering. Since Thomson scattering does not create circular polarization, we will not consider V. It is then convenient to write the polarization as a traceless-symmetric tensor field,

$$P_{ab} = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}. \tag{6}$$

Polarized Boltzmann equation. The Boltzmann equation for the polarized phase space densities is just like that for the intensity: one writes:

$$\frac{Df}{d\eta} = C[f], \tag{7}$$

where $C[f]$ is the collision term, and the derivative $D/d\eta$ transports the 2×2 matrix f according to:

$$\frac{Df_{ab}}{d\eta} = \frac{\partial}{\partial\eta} f_{ab} + \dot{x}^i \frac{\partial}{\partial x^i} f_{ab} + dotp \frac{\partial}{\partial p} f_{ab} + \dot{\hat{p}}^i \frac{\partial}{\partial \hat{p}^i} f_{ab} + h_a^\mu \frac{(\mathbf{u}\cdot\nabla)h_\mu^c}{u^0} h_b^\nu \frac{(\mathbf{u}\cdot\nabla)h_\nu^d}{u^0} f_{ab}. \tag{8}$$

where $a, b \in \{\hat{\theta}, \hat{\phi}\}$ are indices of the 2-dimensional plane perpendicular to direction of photon propagation, i.e. on the unit sphere. In the last term h_a^μ is the 4-vector corresponding to the unit vector in direction a on the unit 2-sphere. This term accounts for the fact that the basis $\{\hat{\theta}, \hat{\phi}\}$ is not parallel-transported along the photon's trajectory, so the polarization can appear to rotate due to the choice of coordinate system. At first order in perturbation theory this effect must vanish since the polarization is first-order and any coordinate rotation is also first-order. A similar argument kills the $\dot{\hat{p}}^i$ term. Also $\dot{x}^i$ multiplies a first-order term (a spatial gradient) so it can be replaced with its zeroeth-order value $\hat{p}^i$. If one is looking at the polarized components, f_{ab} is first-order so one may replace $\dot{p}$ with $-aHp$. Thus the photon Boltzmann equation reduces to:

$$\dot{f}_Q + \hat{p}^i \frac{\partial}{\partial x^i} f_Q - aHp \frac{\partial}{\partial p} f_Q = C[f_Q], \tag{9}$$

and similar for f_U. Writing this in terms of the dimensionless Q and U, we get:

$$-p\frac{\partial f^{(0)}}{\partial p}\dot{Q} - Q\frac{\partial}{\partial\eta}\left(p\frac{\partial f^{(0)}}{\partial p}\right) - p\frac{\partial f^{(0)}}{\partial p}\hat{p}^i\frac{\partial}{\partial x^i}Q + aHp\frac{\partial}{\partial p}\left(p\frac{\partial f^{(0)}}{\partial p}Q\right) = C[f_Q], \tag{10}$$

Since $f^{(0)}$ depends only on the combination ap, the $\partial/\partial\eta$ term cancels against the part of the fourth term where the derivative acts on $p\partial f^{(0)}/\partial p$. Then, after dividing through by $-p\partial f^{(0)}/\partial p$, we get:

$$\dot{Q} + \hat{p}^i \frac{\partial}{\partial x^i} Q + aHpp \frac{\partial f^{(0)}}{\partial p} \frac{\partial}{\partial p} Q = \frac{C[f_Q]}{-p\partial f^{(0)}/\partial p}. \tag{11}$$

This is a remarkably simple equation: note that there is no gravity in it. Polarization is created only by the collision operator, which includes Thomson scattering. Like the case for Θ, it will turn out that Q and U are frequency-independent. We will write the right-hand side as $C[Q]$ so:

$$\dot{Q} + i\mathbf{k}\cdot\hat{\mathbf{p}}Q = C[Q]. \tag{12}$$

Spherical harmonic decomposition. The decomposition of Q and U in spherical harmonics is not as straightforward as Θ because Θ was a scalar whereas Q and U form a tensor. In order for the spherical harmonic modes of Q and U to satisfy the same rotational properties as the Θ_{lm}, we need to construct tensors covariantly derived from the Y_{lm}. The simplest way to do this is to take derivatives of the Y_{lm}'s:

$$Y^E_{lm\ ab}(\theta,\phi) = -\frac{2}{\sqrt{(l-1)l(l+1)(l+2)}}\left(D_aD_b - \frac{1}{2}g_{ab}D^2\right)Y_{lm}, \tag{13}$$

and

$$Y^B_{lm\ ab}(\theta,\phi) = -\frac{1}{\sqrt{(l-1)l(l+1)(l+2)}}\left(\epsilon_{bc}D_aD_c + \epsilon_{ac}D_bD_c\right)Y_{lm}, \tag{14}$$

where D_a is the covariant derivative on the unit sphere and ϵ_{ab} is the Levi-Cevita tensor. These functions form a complete basis for the traceless-symmetric tensor fields on the unit sphere. (One can prove this by applying the second-derivative operators to any polarization field P_{ab} to get a scalar, and then expanding the scalar in spherical harmonics.) It is also possible to show that they differ only by a 45° rotation of the polarization direction. The normalization coefficients have been chosen so that:

$$\int Y^{E*}_{lm\ ab}Y^{E\ ab}_{lm}\, d^2\hat{p}^i = 2, \tag{15}$$

and similarly for B. The 2 is useful because $|f_{ab}|^2$ actually double-counts the square of each Stokes parameter, i.e. it is $2(f_I^2 + f_Q^2 + f_U^2 + f_V^2)$.

Note that there is no $l=0$ or $l=1$ tensor field, which is a consequence of the $e^{\pm 2i\psi}$ dependence of the polarized intensity. One can see this as well by noting that the traceless-symmetric derivative operators applied to Y_{0m} and Y_{1m} give zero.

In addition to rotational properties, which are equivalent to those of Y_{lm}, the tensor spherical harmonics have parity properties:

$$Y_{lm}(\hat{\mathbf{p}}) = (-1)^l Y_{lm}(-\hat{\mathbf{p}}); \quad Y^E_{lm}(\hat{\mathbf{p}}) = (-1)^l Y^E_{lm}(-\hat{\mathbf{p}}); \quad Y^B_{lm}(\hat{\mathbf{p}}) = -(-1)^l Y^B_{lm}(-\hat{\mathbf{p}}). \tag{16}$$

The B-type spherical harmonic has an extra minus sign because its definition included the Levi-Cevita tensor.

It is common to express the polarized phase space density in E and B spherical harmonics:

$$\begin{pmatrix} Q(\hat{\mathbf{p}}) & U(\hat{\mathbf{p}}) \\ U(\hat{\mathbf{p}}) & -Q(\hat{\mathbf{p}}) \end{pmatrix} = \sum_{lm}(-i)^l\sqrt{4\pi(2l+1)}[E_{lm}Y^E_{lm\ ab}(\hat{\mathbf{p}}) + B_{lm}Y^B_{lm\ ab}(\hat{\mathbf{p}})], \tag{17}$$

in analogy to the temperature anisotropies. This equation is the analogue of the spherical harmonic decomposition for polarization. The E_{lm} and B_{lm} transform under rotations in the same way as Θ_{lm}.

Free-streaming term. The polarized Boltzmann equations, Eq. (12), can be transformed to spherical harmonic space by integrating against $Y^{E*}_{lm\,ab}$:

$$\dot{E}_{lm} + \frac{1}{2}k\sum_{l'm'} i^{l+1-l'}\sqrt{\frac{2l'+1}{2l+1}}\int\left[E_{l'm'}Y^{E\,ab}_{l'm'} + B_{l'm'}Y^{B\,ab}_{l'm'}\right]Y^{E*}_{lm\,ab}\cos\theta\,d^2\hat{p}^i$$
$$= \frac{i^l}{2\sqrt{4\pi(2l+1)}}\int P^{ab}Y^{E*}_{lm\,ab}\,d^2\hat{p}^i, \tag{18}$$

and similarly for B_{lm}. The integral on the left-hand side is a free-streaming term: it is the analogue of the integral $\int Y^*_{lm}Y_{l'm'}\cos\theta\,d^2\hat{p}^i$ that we did for the free-streaming of the temperature. Azimuthal symmetry requires $m' = m$ and angular momentum addition requires $l' = l-1, l, l+1$. Also $\cos\theta$ has negative parity (it flips sign if one reverses the direction of $\hat{\mathbf{p}}$) so in the $\dot{E}_{lm}$ equation the $E_{l'm'}$ term enters for $l' = l\pm 1$ and the $B_{l'm'}$ term enters for $l' = l$. The evaluation of the nonzero integrals is tricky; we get:

$$\begin{aligned}\frac{1}{2}\int Y^{E\,ab}_{l'm'}Y^{E*}_{lm\,ab}\cos\theta\,d^2\hat{p}^i &= \delta_{mm'}\delta_{l',l-1}\sqrt{\frac{(l-m)(l+m)(l-2)(l+2)}{l^2(2l-1)(2l+1)}}\\ &\quad+\delta_{mm'}\delta_{l',l+1}\sqrt{\frac{(l+1-m)(l+1+m)(l-1)(l+3)}{(l+1)^2(2l+1)(2l+3)}}\end{aligned} \tag{19}$$

and

$$\frac{1}{2}\int Y^{B\,ab}_{l'm'}Y^{E*}_{lm\,ab}\cos\theta\,d^2\hat{p}^i = \delta_{mm'}\delta_{ll'}\frac{2m}{l(l+1)}. \tag{20}$$

We then conclude that:

$$\begin{aligned}\dot{E}_{lm} &= \frac{\sqrt{(l-2)(1+2)(l-m)(l+m)}}{l(2l+1)}kE_{l-1,m} - \frac{\sqrt{(l-1)(l+3)(l+1-m)(l+1+m)}}{(l+1)(2l+1)}kE_{l+1,m}\\ &\quad-\frac{2im}{l(l+1)}kB_{lm} + C[E_{lm}].\end{aligned} \tag{21}$$

A similar equation holds for B:

$$\begin{aligned}\dot{B}_{lm} &= \frac{\sqrt{(l-2)(1+2)(l-m)(l+m)}}{l(2l+1)}kB_{l-1,m} - \frac{\sqrt{(l-1)(l+3)(l+1-m)(l+1+m)}}{(l+1)(2l+1)}kB_{l+1,m}\\ &\quad+\frac{2im}{l(l+1)}kE_{lm} + C[B_{lm}].\end{aligned} \tag{22}$$

Collision operator. The collision term for polarization can depend only on the local phase space density of the photons, and only on the photon density at that particular frequency. Symmetry (rotation, parity) considerations also dictate that the collision term $C[E_{lm}]$ can depend only on quantities with the same angular momentum and parity, i.e. Θ_{lm} and E_{lm}, whereas $C[B_{lm}]$ can depend only on B_{lm}. Also the coefficients can depend only on l and not m:

$$C[E_{lm}] = \dot{\tau}E_{lm} - \dot{\tau}(\alpha_l\Theta_{lm} + \beta_l E_{lm});\quad C[B_{lm}] = \dot{\tau}B_{lm} - \dot{\tau}\gamma_l B_{lm}. \tag{23}$$

Our job is to derive the coefficients α_l, β_l, and γ_l. Note that we have taken the term associated with removal of photons out front and the α_l, β_l, and γ_l then describe radiation that has been re-emitted after scattering.

The simplest way to do this is to consider the $m = 0$ case. Because of the ϕ-independence of Y_{l0}, the E-type spherical harmonics $\mathbf{Y}^E_{lm}$ have only Q polarization and the B-types have only U. Thus we may derive α and β by taking an $m = 0$ mode and keeping only the Q polarization. There is then a separate vertical (north-south) temperature perturbation Θ_V and a horizontal (east-west) perturbation Θ_H, given by

$$\Theta_V = \Theta + Q, \quad \Theta_H = \Theta - Q. \tag{24}$$

One can estimate the post-scattering intensity and polarization by looking at the polarization-resolved differential scattering cross section from initial direction/polarization $\hat{\mathbf{p}}', \zeta'$ to $\hat{\mathbf{p}}, \zeta$. The part of the collision term due to re-scattered radiation is:

$$C[\Theta_V(\hat{\mathbf{p}})]_{\rm rescat} = |\dot{\tau}| \int \left[\frac{dP_{V\to V}}{d\Omega} \Theta_V(\hat{\mathbf{p}}') + \frac{dP_{H\to V}}{d\Omega} \Theta_H(\hat{\mathbf{p}}') \right] d^2\hat{\mathbf{p}}'. \tag{25}$$

The polarization-resolved differential probability is

$$\frac{dP_{\zeta'\to\zeta}}{d\Omega}(\hat{\mathbf{p}}' \to \hat{\mathbf{p}}) = \frac{3}{8\pi}(\zeta\cdot\zeta')^2. \tag{26}$$

Now for directions θ', ϕ' and θ, ϕ, one may take the horizontal and vertical polarization vectors:

$$\zeta_V = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta); \quad \zeta_H = (-\sin\phi, \cos\phi, 0), \tag{27}$$

and then the differential probabilities are:

$$\begin{aligned}
\frac{dP_{H\to H}}{d\Omega}(\hat{\mathbf{p}}' \to \hat{\mathbf{p}}) &= \frac{3}{8\pi}\cos^2\Delta\phi, \\
\frac{dP_{H\to V}}{d\Omega}(\hat{\mathbf{p}}' \to \hat{\mathbf{p}}) &= \frac{3}{8\pi}\cos^2\theta\sin^2\Delta\phi, \\
\frac{dP_{V\to H}}{d\Omega}(\hat{\mathbf{p}}' \to \hat{\mathbf{p}}) &= \frac{3}{8\pi}\cos^2\theta'\sin^2\Delta\phi, \\
\frac{dP_{V\to V}}{d\Omega}(\hat{\mathbf{p}}' \to \hat{\mathbf{p}}) &= \frac{3}{8\pi}(\cos\theta\cos\theta'\cos\Delta\phi + \sin\theta\sin\theta')^2.
\end{aligned} \tag{28}$$

where $\Delta\phi = \phi - \phi'$. Since we're looking at $m = 0$ modes, one may average over $\Delta\phi$, and then get:

$$\begin{aligned}
C[\Theta_V(\hat{\mathbf{p}})]_{\rm rescat} &= \frac{3}{4}|\dot{\tau}| \int \left[(\frac{1}{2}\cos^2\theta\cos^2\theta' + \sin^2\theta\sin^2\theta')\Theta_V(\mathbf{p}') + \frac{1}{2}\cos^2\theta\Theta_H(\mathbf{p}') \right] \sin\theta'\, d\theta'; \\
C[\Theta_H(\hat{\mathbf{p}})]_{\rm rescat} &= \frac{3}{4}|\dot{\tau}| \int \left[\frac{1}{2}\cos^2\theta'\Theta_V(\mathbf{p}') + \frac{1}{2}\Theta_H(\mathbf{p}') \right] \sin\theta'\, d\theta'.
\end{aligned} \tag{29}$$

Using $Q = (\Theta_V - \Theta_H)/2$, one may convert this into a scattering term for Q:

$$\begin{aligned} C[Q(\hat{\mathbf{p}})]_{\text{rescat}} &= \frac{3}{8}|\dot{\tau}| \int \left[\sin^2\theta(\sin^2\theta' - \frac{1}{2}\cos^2\theta')\Theta_V(\mathbf{p}') - \frac{1}{2}\sin^2\theta\Theta_H(\mathbf{p}')\right]\sin\theta'\,d\theta' \\ &= \frac{3}{8}\sin^2\theta|\dot{\tau}|\int[(\frac{1}{2} - \frac{3}{2}\cos^2\theta')\Theta(\mathbf{p}') + \frac{3}{2}\sin^2\theta' Q(\mathbf{p}')]\sin\theta'\,d\theta'. \end{aligned} \tag{30}$$

This equation can be simplified if we recall the values of the spherical harmonics,

$$Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1), \quad Y^E_{20\,\hat{\theta}\hat{\theta}} = \sqrt{\frac{15}{32\pi}}\sin^2\theta. \tag{31}$$

It is clear from inspection of $C[Q]_{\text{rescat}}$ that only these two spherical harmonics are involved. In particular, if we convert to lm space, $C[E_{lm}]$ vanishes unless $l = 2$. For this special case, we get (by converting to spherical harmonics and using orthonormality):

$$C[E_{20}]_{\text{rescat}} = \left(-\frac{\sqrt{6}}{10}\Theta_{20} + \frac{3}{5}E_{20}\right)|\dot{\tau}|. \tag{32}$$

That is, $\alpha_2 = -\sqrt{6}/10$ and $\beta_2 = 3/5$. By spherical symmetry this equation must apply to the other values of m.

A similar calculation can be done for B and shows the re-scattering term to be zero, $\gamma_l = 0$.

Thus the overall system of equations for polarization is:

$$\begin{aligned} \dot{E}_{lm} &= \frac{\sqrt{(l-2)(1+2)(l-m)(l+m)}}{l(2l+1)}kE_{l-1,m} - \frac{\sqrt{(l-1)(l+3)(l+1-m)(l+1+m)}}{(l+1)(2l+1)}kE_{l+1,m} \\ &\quad -\frac{2im}{l(l+1)}kB_{lm} - |\dot{\tau}|\left(E_{lm} + \frac{\sqrt{6}\Theta_{2m} - 6E_{2m}}{10}\delta_{l2}\right). \end{aligned} \tag{33}$$

and

$$\begin{aligned} \dot{B}_{lm} &= \frac{\sqrt{(l-2)(1+2)(l-m)(l+m)}}{l(2l+1)}kB_{l-1,m} - \frac{\sqrt{(l-1)(l+3)(l+1-m)(l+1+m)}}{(l+1)(2l+1)}kB_{l+1,m} \\ &\quad +\frac{2im}{l(l+1)}kE_{lm} - |\dot{\tau}|B_{lm}. \end{aligned} \tag{34}$$

Power spectra. In the case of the CMB temperature, we defined a power spectrum based on the local decomposition of the temperature anisotropy into spherical harmonics:

$$\Theta(\hat{\mathbf{p}}) = \sum_{lm} a_{lm}Y_{lm}(\hat{\mathbf{p}}), \quad \langle a^*_{lm}a_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}. \tag{35}$$

For polarization one can define a similar decomposition:

$$\begin{pmatrix} Q(\hat{\mathbf{p}}) & U(\hat{\mathbf{p}}) \\ U(\hat{\mathbf{p}}) & -Q(\hat{\mathbf{p}}) \end{pmatrix} = \sum_{lm} \left[a^E_{lm} Y^E_{lm\;\hat{a}\hat{b}}(\hat{\mathbf{p}}) + a^B_{lm} Y^B_{lm\;\hat{a}\hat{b}}(\hat{\mathbf{p}}) \right]. \tag{36}$$

There are separate E-type and B-type power spectra,

$$\langle a^{E*}_{lm} a^E_{l'm'} \rangle = C^{EE}_l \delta_{ll'} \delta_{mm'}, \quad \langle a^{B*}_{lm} a^B_{l'm'} \rangle = C^{BB}_l \delta_{ll'} \delta_{mm'}. \tag{37}$$

Because E has the same parity as temperature, it is also possible to have a temperature-polarization spectrum:

$$\langle a^*_{lm} a^E_{l'm'} \rangle = C^{\Theta E}_l \delta_{ll'} \delta_{mm'}. \tag{38}$$

The power spectra for the scalars can be determined using the same method as for the temperature, i.e. integrating over wavenumbers and angles to get:

$$\begin{aligned} C^{EE}_l &= 4\pi \int \Delta^2_\zeta(k) \left| \frac{E_l}{\zeta}(k) \right|^2 d\ln k \\ C^{\Theta E}_l &= 4\pi \int \Delta^2_\zeta(k) \Re \left[\frac{\Theta^*_l}{\zeta}(k) \frac{E_l}{\zeta}(k) \right] d\ln k. \end{aligned} \tag{39}$$

The real part and complex conjugate are technically unnecessary for the scalars since Θ_l/ζ and E_l/ζ are real.

2 Phenomenology

A few results follow easily from the above equations:

- Thomson scattering of the local quadrupole Θ_{2m} of the temperature field is the only source for polarization.
- Generation of polarization can only happen in regions where the optical depth is high enough to have Thomson scattering but not so high as to wash out the quadrupole. The two such possibilities are the recombination surface and reionization.
- Thomson scattering can only generate $l = 2$ E-type polarization; the free-streaming terms are needed to generate everything else.
- The mixing of E into B-type polarization occurs via a single term in the $\dot{B}_{lm}$ equation that has a factor of m. Therefore for scalar perturbations ($m = 0$) there is no way to generate B-type polarization. On the other hand tensors can generate it.

Recombination epoch. Polarization can be generated at recombination because of the finite thickness of the last scattering surface. The finite optical depth $|\dot{\tau}| < \infty$ allows a Θ_2 to be generated, and then converted into polarization.

We can get an approximate sense for the magnitude of the polarization by using the tight-coupling limit:

$$\dot{\Theta}_2 = \frac{2}{5}k\Theta_1 + \frac{9}{10}\dot{\tau}\Theta_2, \tag{40}$$

and supposing that the two terms on the right side approximately balance, which should be true if $\dot{\tau}$ is large. Then:

$$\Theta_2 \approx \frac{4k}{9|\dot{\tau}|}\Theta_1. \tag{41}$$

The polarization generated by the source term is:

$$\dot{E}_2 \approx -\frac{\sqrt{6}}{10}|\dot{\tau}|\Theta_2 \approx -\frac{2\sqrt{6}}{45}k\Theta_1. \tag{42}$$

Thus right after recombination we should have a polarization field of

$$E_2(\eta_{rec}) \approx -\frac{2\sqrt{6}}{45}k\Delta\eta_{lss}\Theta_1, \tag{43}$$

where $\Delta\eta_{lss}$ is the width of the last scattering surface, i.e. the time during which the above equations are valid. But Θ_1 is an oscillating function; in the small-scale limit we have

$$\Theta_1 \approx -\frac{\zeta}{\sqrt{3}}\sin\frac{k\eta_{rec}}{\sqrt{3}}\ e^{-k^2/k_D^2}, \tag{44}$$

so

$$E_2(\eta_{rec}) \approx \frac{2\sqrt{2}}{45}k\Delta\eta_{lss}\sin\frac{k\eta_{rec}}{\sqrt{3}}\ e^{-k^2/k_D^2}\zeta. \tag{45}$$

This is an oscillating function, which rapidly goes to zero at large k, and also has an exponential cutoff. It is proportional to the width of the surface of last scattering. It is smaller than Θ_1 by a factor of $k\Delta\eta_{lss}$.

In order to determine what the polarization looks like today we need to do a radiative transfer calculation. This is analogous to the spherical Bessel function calculation for temperature, except that the polarization equations are more complicated and the solution is a tensor spherical Bessel function. Qualitatively, however, the results are similar to those for temperature: the power spectrum C_l^{EE} today is an integral over $\Delta_\zeta^2(k)$ weighted heavily at $k \sim lr_{CMB}$. Because E_2 has a sine instead of a cosine dependence, the results are:

- The E-type polarization power spectrum C_l^{EE} shows acoustic oscillations, but 180° out of phase with $C_l^{\Theta\Theta}$ ($\sin^2$ vs. $\cos^2$).
- The cross-correlation $C_l^{\Theta E}$ is 90° out of phase with both ($\sin\cos$).

Reionization epoch. Theory predicts that the universe should have become neutral at $z \sim 1200$ and the existence of acoustic oscillations confirms that this picture is basically correct. However we know that the universe must have become reionized again from studies of hydrogen (Lyman-α) absorption lines in quasars. A neutral intergalactic medium would present an optical depth of $\sim 10^4$ and all flux blueward of Lyman-α in the quasar rest frame would be wiped out. Instead what is observed is a complex series of absorption features whose fractional transmission increases with redshift, being about 50% at $z \sim 3$ and becoming complete above $z = 6$. This implies that some mechanism reionized almost all of the gas in the universe some time before $z = 6$. The most likely candidate is UV radiation from an early generation of stars.

Reionization causes an additional source of optical depth between us and the recombination surface. If reionization were a step function at $z = z_{ri}$, with post-reionization electron abundance x_e, then this optical depth is

$$\tau_{ri} = \int n_e \sigma_T dt = \frac{\sigma_T n_{H0} x_e}{H_0} \int_0^{z_{ri}} (1+z)^3 \frac{dz}{(1+z)\sqrt{\Omega_\Lambda + \Omega_m (1+z)^3}}. \tag{46}$$

If z_{ri} is large then the cosmological constant has only a minor influence; removing it reduces the integral to:

$$\tau_{ri} \approx \frac{2\sigma_T n_{H0} x_e}{3\Omega_m^{1/2} H_0} (1+z_{ri})^{3/2}. \tag{47}$$

The usual assumption is that at reionization, H became ionized to H^+ and He to He^+, which gives $x_e = 1.08$ (1 electron from H and 0.08 from He). (He$\to$He$^+$ is predicted by simulations to occur at the same time as H$\to$H$^+$ because of the spectrum of the stars.) Under these conditions, we get:

$$\tau_{ri} \approx 0.0023(1+z_{ri})^{3/2}. \tag{48}$$

The requirement of $z_{ri} > 6$ from the quasar absorption features implies $\tau_{ri} > 0.043$.

In order to measure τ_{ri} we must understand its impact on the CMB. If one studies the Boltzmann equation, one can see that all of the high multipoles in the CMB have terms in the $\dot{\Theta}_l$ equation that contain $-|\dot{\tau}|\Theta_l$ for $l \geq 1$. These terms become inactive after recombination, but turn on again due to reionization. This implies that all of the modes that were inside the horizon at reionization (hence have temperature anisotropies dominated by large l) are suppressed by a factor of $\exp(-\tau_{ri})$. The condition for this to occur is roughly $k\eta_{ri} \gg 1$, or

$$l = k(\eta_0 - \eta_{ri}) \gg \frac{\eta_0 - \eta_{ri}}{\eta_{ri}}. \tag{49}$$

Within this range the power spectrum, which depends on Θ_l^2, is suppressed by a factor:

$$C_l \to C_l \exp(-2\tau_{ri}). \tag{50}$$

Therefore reionization causes a suppression of all the high multipoles. This makes sense: the additional scattering wipes out small-scale structure.

A second effect of reionization is on the CMB polarization. For modes that were outside the horizon at recombination, we found in the temperature anisotropy section that

$$\Theta_l(\eta) = -\frac{1}{5}\zeta j_l(k\eta), \tag{51}$$

and in particular

$$\Theta_2(\eta_{ri}) = -\frac{1}{5}\zeta j_2(k\eta_{ri}). \tag{52}$$

If the scattering from reionization were instantaneous (it's not) then this immediately generates E-type polarization:

$$E_2(\eta_{ri}) = \frac{\sqrt{6}}{50}\tau_{ri}\zeta j_2(k\eta_{ri}). \tag{53}$$

The free-streaming converts this into a polarization today at $l \sim k(\eta_0 - \eta_{ri})$. Since the spherical Bessel function is dominated by arguments near ~ 3, we thus expect E-mode polarization to peak at

$$l \sim 3\frac{\eta_0 - \eta_{ri}}{\eta_{ri}} \tag{54}$$

and have an amplitude proportional to τ_{ri}. In the power spectrum one expects:

$$C_l^{EE} \propto \tau_{ri}^2; \quad C_l^{\Theta E} \propto \tau_{ri}. \tag{55}$$

The expected polarization per $\ln l$, $\sqrt{l(l+1)C_l^{EE}/2\pi}$ is of order $\tau_{ri}\Delta_\zeta \sim 10^{-6}$, i.e. at the microKelvin level; the factor of $\sqrt{6}/50$ makes this even lower. Nevertheless this polarization feature at low l it was detected by WMAP, which finds $\tau_{ri} = 0.087 \pm 0.017$ and $z_{ri} = 11.0 \pm 1.4$.

Gravitational waves. Primordial gravitational waves are expected to be very weak, and their imprint on the CMB temperature fluctuations would be very hard to disentangle from that of the density fluctuations. In polarization however they have a unique signature: the B-type polarization, which is not generated by density perturbations.

The formal way to compute the B-type polarization signature is to superpose many waves:

$$C_l^{BB} = 8\pi \int \Delta_h^2(k) \left|\frac{B_l}{h}(k)\right|^2 d\ln k. \tag{56}$$

We may however make an educated estimate as follows. The rate of generation of photon quadrupole at the recombination surface is:

$$\dot{\Theta}_2 \sim -\frac{\dot{E}}{5} \equiv \frac{2}{5\sqrt{3}}\dot{h}, \tag{57}$$

where we have written the gravitational wave amplitude in terms of $h = (h_+ \mp ih_\times)/\sqrt{2}$ instead of E to avoid confusion with the polarization. This occurs throughout the time of last scattering, so during this surface Θ_2 is of order

$$\Theta_2 \sim \frac{2}{5\sqrt{3}}\dot{h}\Delta\eta_{lss}. \tag{58}$$

Thomson scattering then generates E-type polarization:

$$\dot{E}_2 \sim -\frac{\sqrt{6}}{10}|\dot{\tau}|\Theta_2 \sim -\frac{\sqrt{2}}{25}\dot{h}\Delta\eta_{lss}|\dot{\tau}|. \tag{59}$$

Using $|\dot{\tau}| \sim 1/\Delta\eta_{lss}$ and integrating over the surface of last scatter, we get:

$$E_2 \sim -\frac{\sqrt{2}}{25}\dot{h}\Delta\eta_{lss}. \tag{60}$$

The dominant gravitational wave modes will be those that enter the horizon at recombination $k \sim 1/\eta_{rec}$: modes that enter earlier have adiabatically decayed away, and those that enter later have $h \neq 0$ but $\dot{h} \approx 0$. For these waves, $\dot{h} \sim h/\eta_{rec}$, and the typical polarization is:

$$\frac{\sqrt{2}}{25}\frac{\Delta\eta_{lss}}{\eta_{rec}}h_{\mathrm{rms}} \sim 10^{-7}r^{1/2}. \tag{61}$$

This is at a scale of $l \sim k\eta_0 \sim \eta_0/\eta_{rec} \sim 50$, i.e. a few degrees. However for typical models with $r \sim 0.1$ the amplitude is down in the range of 100 nanoKelvins, and a more careful calculation gives a somewhat lower number. This makes gravitational waves one of the most difficult problems in observational cosmology. Nevertheless there is an enormous prize: measuring r and hence setting the energy scale of the inflationary epoch.

Gravitational waves also generate polarization at reionization, however this is on the largest scales (l of a few) where foregrounds (see below) are most severe.

3 Systematics

No discussion of the CMB would be complete without a brief mention of the problems facing observers who measure such tiny signals. Here we give an incomplete list:

- *The ground*: The CMB polarization fluctuations are a few μK, but the ground is at $\sim$ 300 K. Therefore if even a small amount of ground radiation diffracts into the telescope it is a serious problem. Ground-based experiments must take care to minimize diffraction, and also take advantage of the fact that the sky rotates relative to the ground so that the two effects can be separated. Going to space also helps but is expensive.

- *Atmosphere*: The Earth's atmosphere contains H_2O and O_2 molecules that radiate in the microwave bands. Humidity variations can masquerade as CMB anisotropies. These move relative to the sky and do not repeat from day to day, so once again there are ways to separate them, nevertheless they are so large that they must be very carefully removed. Balloon or space experiments have an advantage as they rise above most of the water.

- *Beams*: Precise measurement of the CMB fluctuations requires that one understand the beam (i.e. how the response to a source varies depending on how far it is off the boresight) very well. These are usually determined by diffraction: the resolution of an experiment is no better than $\theta = \lambda/D$. But to measure CMB power spectrum to 1%, we need to know the Fourier transform of the beam to 0.5%. Often one uses a bright microwave source such as a planet for this purpose.

- *Intensity-to-polarization leakage*: Since the CMB temperature fluctuations are much brighter than polarization one must make sure that the two polarizations measured by the instrument have the same relative calibration and that features such as polarized diffraction spikes are well understood. The CMB temperature fluctuations are much fainter than the ground, so one might think they are less of a problem; but they are fixed to the sky which may make them more pernicious than ground pick-up.

- *Response to magnetic fields*: The Earth's magnetic field can affect some types of microwave detectors, especially those using SQUIDs to measure current. These must be carefully shielded using superconducting cages.

There are also foregrounds: objects that emit microwaves that are not the CMB.

- *Active galactic nuclei*: These emit synchrotron radiation that is often time-dependent. They have a different spectrum than the CMB, tilted to lower frequencies than a blackbody. The brighter ones can be recognized easily in CMB maps and are usually (though not always) pointlike but the fainter ones may not. Some experiments, including WMAP, must do a statistical subtraction of AGN.

- *Star-forming galaxies*: These emit synchrotron, free-free radiation, and also thermal radiation from dust grains that have been heated by absorption of starlight. They are much fainter than AGN but with several emission mechanisms may have complex spectra. To date they have not been a problem but the next generation of higher-frequency CMB experiments (≥ 150 GHz) could face significant difficulties, especially at small angular scales.

- *Galactic synchrotron*: Our own Milky Way emits synchrotron radiation, which fills the entire sky and at low frequencies (22 GHz) contributes tens of μK even at high Galactic latitude. Synchrotron is highly polarized which makes it a special problem for CMB experiments. It is steeply frequency-dependent, being much brighter at low frequency, so maps at e.g. 400 MHz are often used to assess contamination.

- *Galactic free-free radiation*: This is present but not the dominant foreground at any frequency. It has a well-understood spectral dependence, $I_\nu \propto \nu^{-0.15}$, so it is most important at low frequency. It is intrinsically unpolarized in the optically thin regime, and its source, warm ionized gas, is also traced at optical wavelengths by diffuse Hα emission.

- *Galactic dust*: Interstellar dust absorbs starlight and can re-radiate it at infrared wavelengths; a small fraction of the energy emerges in the microwave via the Rayleigh-Jeans tail. The emission is weakly ($\sim 5\%$) polarized due to alignment of the dust grains with the magnetic field. There is also evidence for an additional dust emission process, possibly electric dipole radiation from spinning dust grains, or thermal fluctuations of the magnetic moment of iron-bearing grains. The distribution of dust can be estimated based on the 100 μm maps of the sky from the IRAS satellite. Thermal dust emission dominates the Galactic foreground above $\sim$80 GHz.

The foregrounds are a serious problem, but by rejecting data from the Galactic plane (where they are worst), using their frequency dependence, and incorporting data from other wavelengths, they have so far been overcome. They will however represent a major challenge, especially for gravitational wave detection.