1 Primary CMB anisotropies

We finally come to the determination of the CMB anisotropy power spectrum. This set of lectures will be divided into five parts:

- CMB power spectrum formalism.
- Radiative transfer: from recombination to today.
- Large scales: Sachs-Wolfe effect.
- Intermediate scales: Acoustic peaks.
- Small scales: Damping tail.

We will only consider the scalar perturbations here as they dominate the CMB power spectrum, but we'll consider the tensors as well when we do polarization.

2 Formalism

The CMB, like any field on the unit sphere, is conveniently decomposed in spherical harmonics. That is,

$$\Theta(\mathbf{x}, \hat{\mathbf{p}}, \eta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(\mathbf{x}, \eta) Y_{lm}(\hat{\mathbf{p}}). \tag{1}$$

We measure the CMB at one particular point in the Universe, which we conveniently take to be the origin $\mathbf{x} = 0$. Therefore our job is to understand the statistics of the local multipole moments $a_{lm}(0, \eta_0)$.

To do this, let's begin by considering the Fourier expansion of Θ:

$$\Theta(\mathbf{x}, \hat{\mathbf{p}}, \eta) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Theta(\mathbf{k}, \hat{\mathbf{p}}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2}$$

At $\mathbf{x} = 0$, the last exponential vanishes. We can also replace the Fourier-space Θ with its multipole moments:

$$\Theta(\mathbf{k}, \hat{\mathbf{p}}, \eta) = \sum_{l} (-i)^l \sqrt{4\pi(2l+1)}\, \Theta_l(\mathbf{k}, \eta) Y_{l0}(\theta', \phi'), \tag{3}$$

where θ', ϕ' are the coordinates of $\hat{\mathbf{p}}$ in the frame rotated to put $\mathbf{k}$ on the 3-axis. That is,

$$\cos\theta' = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}. \tag{4}$$

(Here ϕ' doesn't matter because we only considered scalars.) Since the spherical harmonics are related to Legendre polynomials via:

$$Y_{l0}(\theta', \phi') = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta') = \sqrt{\frac{2l+1}{4\pi}} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}), \tag{5}$$

we have:

$$\Theta(\mathbf{k},\hat{\mathbf{p}},\eta)=\sum_l(-i)^l(2l+1)\Theta_l(\mathbf{k},\eta)P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}). \tag{6}$$

Substituting into the expression for real-space Θ, we get:

$$\Theta(0,\hat{\mathbf{p}},\eta)=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_l(-i)^l(2l+1)\Theta_l(\mathbf{k},\eta)P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}). \tag{7}$$

In order to get the a_{lm}'s, we use the spherical harmonic addition theorem,

$$P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{p}})Y^*_{lm}(\hat{\mathbf{k}}), \tag{8}$$

so that:

$$\Theta(0,\hat{\mathbf{p}},\eta)=4\pi\sum_{lm}\int\frac{d^3\mathbf{k}}{(2\pi)^3}(-i)^l\Theta_l(\mathbf{k},\eta)Y_{lm}(\hat{\mathbf{p}})Y^*_{lm}(\hat{\mathbf{k}}). \tag{9}$$

We can then read off the local multipole moments:

$$a_{lm}(0,\eta)=4\pi\int\frac{d^3\mathbf{k}}{(2\pi)^3}(-i)^l\Theta_l(\mathbf{k},\eta)Y^*_{lm}(\hat{\mathbf{k}}). \tag{10}$$

The average of any of the a_{lm}'s will be zero because our initial perturbations are equally likely to be positive or negative. Therefore we want their second moment:

$$\langle a_{lm}a^*_{l'm'}\rangle=16\pi^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{d^3\mathbf{k}'}{(2\pi)^3}i^{l'-l}\langle\Theta_l(\mathbf{k})\Theta^*_{l'}(\mathbf{k}')\rangle Y^*_{lm}(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{k}}'). \tag{11}$$

Let's focus on the expectation value of products of Θs here. In linear perturbation theory, both are determined by the primordial curvature perturbation:

$$\langle\Theta_l(\mathbf{k})\Theta^*_{l'}(\mathbf{k}')\rangle=\frac{\Theta_l(\mathbf{k})}{\zeta(\mathbf{k})}\frac{\Theta^*_{l'}(\mathbf{k})}{\zeta^*(\mathbf{k}')}\langle\zeta(\mathbf{k})\zeta^*(\mathbf{k}')\rangle. \tag{12}$$

The latter expectation value is just

$$\langle\zeta(\mathbf{k})\zeta^*(\mathbf{k}')\rangle=(2\pi)^3P_\zeta(k)\delta(\mathbf{k}-\mathbf{k}'); \tag{13}$$

recall that was the definition of the power spectrum. Plugging it all in yields:

$$\langle a_{lm}a^*_{l'm'}\rangle=16\pi^2\int\frac{d^3\mathbf{k}}{(2\pi)^3}i^{l'-l}\frac{\Theta_l(\mathbf{k})}{\zeta(\mathbf{k})}\frac{\Theta^*_{l'}(\mathbf{k})}{\zeta^*(\mathbf{k})}P_\zeta(k)Y^*_{lm}(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{k}}). \tag{14}$$

The next step is to separate the radial integral over k from the angular integral over $\hat{\mathbf{k}}$:

$$\int\frac{d^3\mathbf{k}}{(2\pi)^3}\to\int\frac{k^3}{2\pi^2}\,d\ln k\int\frac{d^2\hat{\mathbf{k}}}{4\pi}. \tag{15}$$

The physics is encoded in the ratio $\Theta_l(\mathbf{k})/\zeta(\mathbf{k})$ which is direction-independent, whereas the spherical harmonics depend only on $\hat{\mathbf{k}}$:

$$\langle a_{lm} a^*_{l'm'} \rangle = 4\pi \int \frac{k^3}{2\pi^2}\, d\ln k\, i^{l'-l} \frac{\Theta_l(k)}{\zeta(k)} \frac{\Theta^*_{l'}(k)}{\zeta^*(k)} P_\zeta(k) \int d^2\hat{\mathbf{k}}\, Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}). \tag{16}$$

The last integral is $\delta_{ll'}\delta_{mm'}$ by orthonormality of the spherical harmonics. This collapses the rest of the integral to yield:

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'}\delta_{mm'}\ 4\pi \int \Delta^2_\zeta(k) \left| \frac{\Theta_l(k)}{\zeta(k)} \right|^2 d\ln k. \tag{17}$$

Therefore all of the a_{lm}s are uncorrelated, and their variance depends only on l, and not m. This is a consequence of rotational symmetry, which here was encoded in the fact that the power spectra $P_\zeta(k)$ and the evolution equations depend only on the magnitude of $\mathbf{k}$, not its direction. We express this as:

$$\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}, \tag{18}$$

where C_l is the *angular power spectrum.* It is given by:

$$C_l = 4\pi \int \Delta^2_\zeta(k) \left| \frac{\Theta_l(k)}{\zeta(k)} \right|^2 d\ln k. \tag{19}$$

This is the quantity (or set of quantities) that most CMB observers aim to measure.

We can associate each l with a typical angular size (half-wavelength),

$$\theta \sim \frac{\pi}{l} = \frac{180^\circ}{l}, \tag{20}$$

but this is only a heuristic correspondance. When looking at the surface of last scattering at $z = 1000$, we can associate this with a comoving length scale,

$$x \sim r_{CMB}\theta \sim \frac{\pi r_{CMB}}{l}, \tag{21}$$

and hence with a wavenumber

$$k \sim \frac{\pi}{x} \sim \frac{l}{r_{CMB}}, \tag{22}$$

but again these equations are only indicative of the order of magnitude of the wavenumber k that dominates a particular multipole l. As one can see from Eq. (19), each l actually comes from a range of k's.

Usually one starts measuring the power spectrum at $l = 2$. This is because the monopole $l = 0$ is unobservable: we don't know the "true" mean value of the CMB temperature, only its value at our location. (It is also gauge dependent.) The dipole $l = 1$ depends on the observer's velocity, so it has no absolute meaning. It does make sense to ask what is the CMB dipole in a frame that is at rest relative to distant galaxies, but in practice the latter frame is not known well enough to make this exercise useful. The $l \geq 2$ multipoles, measured in the rest frame of the CMB, are well-defined and gauge-invariant.

The power spectrum as defined here is dimensionless, but some CMB observers prefer to quote a_{lm} in μK, i.e. they report perturbations as ΔT rather than fractional perturbations $\Delta T/T$. In this case all formulas for C_l should be multiplied by $T_{\gamma 0}^2$, where $T_{\gamma 0} = 2.73 \times 10^6\,\mu$K.

3 Radiative transfer: from recombination to today.

In this section, we consider what happens to photons after recombination. We clearly need this to compute Θ_l and hence the CMB power spectrum.

In the absence of scattering, the photon multipoles equations are:

$$\begin{aligned}
\dot{\Theta}_0 &= -k\Theta_1 - \dot{\Phi}; \\
\dot{\Theta}_1 &= \frac{k}{3}(\Theta_0 - 2\Theta_2) + \frac{1}{3}k\Psi; \\
\dot{\Theta}_l &= \frac{k}{2l+1}[l\Theta_{l-1} - (l+1)\Theta_{l+1}]. \qquad (23)
\end{aligned}$$

These equations simplify if we note that $\Psi = -\Phi$ once the photons and neutrinos don't contribute significantly to the energy density, and define:

$$\bar{\Theta}_0 \equiv \Theta_0 + \Psi = \Theta_0 - \Phi; \quad \bar{\Theta}_l \equiv \Theta_l \ (l \geq 1). \qquad (24)$$

The system of equations reduces to:

$$\begin{aligned}
\dot{\bar{\Theta}}_0 &= -k\bar{\Theta}_1 - 2\dot{\Phi}; \\
\dot{\bar{\Theta}}_l &= \frac{k}{2l+1}[l\bar{\Theta}_{l-1} - (l+1)\bar{\Theta}_{l+1}] \ \ (l \geq 1). \qquad (25)
\end{aligned}$$

Since the Universe was optically thick (large $\dot{\tau}$) prior to recombination, we will have $\Theta_l(\eta_{rec}) = 0$ for $l \geq 2$. Thus the solutions to Eq. (25) are determined by the initial conditions $\bar{\Theta}_{0,1}(\eta_{rec})$ and the metric source $\dot{\Phi}(\eta)$. Since the equation is linear, these three contributions can be assessed separately.

Initial monopole perturbation. Let's suppose first that at recombination $\bar{\Theta}_0 = 1$ and $\bar{\Theta}_l = 0$ $(l \geq 1)$, and that $\dot{\Phi} = 0$. From the derivative relation for the spherical Bessel functions,

$$(2l+1)j_l'(x) = l j_{l-1}(x) - (l+1)j_{l+1}(x), \qquad (26)$$

we can see that the solution is

$$\bar{\Theta}_l = j_l(k\Delta\eta), \quad \Delta\eta = \eta - \eta_{rec}. \qquad (27)$$

This is not surprising: since in this approximation the photons are simply free-streaming, one would expect that at $k\Delta\eta < 1$ an observer should see only a monopole, but at late times an observer sees many ($\sim \Delta\eta/\lambda = k\Delta\eta/2\pi$) perturbation wavelengths. In this case the dominant multipole is

$$l \sim 2\pi[\text{numberofwavesperradian}] \sim k\Delta\eta. \tag{28}$$

Since $j_l(x)$ peaks at $x \sim l$ this is indeed what we get.

Initial dipole perturbation. Now let's suppose that initially $\bar{\Theta}_1 = 1$, and all other $\bar{\Theta}_l$ vanish. Also suppose $\dot{\Phi} = 0$ at all times. Since the equations of motion are η-independent, if $j_l(k\Delta\eta)$ is a solution, then the derivative with respect to η of this is also a solution. That is, one solution is:

$$\bar{\Theta}_l = 3j_l'(k\Delta\eta). \tag{29}$$

This is the solution that satisfies our initial conditions; recall that at small x, $j_l(x) \to x^l/(2l+1)!!$.

At large l this is very similar to the monopole solution, except that there is a factor of 3, and that the $'$ implies that the phase of oscillation is $90°$ out of phase from the monopole.

Time-varying potentials. We take a Green's function approach. If initially $\bar{\Theta}_l = 0$, but $\dot{\Phi}$ is a delta function at some conformal time η_1,

$$\dot{\Phi} = \delta(\eta_1), \tag{30}$$

then immediately afterward $\bar{\Theta}_0 = -2$, and the subsequent evolution is:

$$\bar{\Theta}_l = -2j_l(k(\eta - \eta_1)). \tag{31}$$

We can get the evolution for general $\dot{\Phi}$ by superposition:

$$\bar{\Theta}_l = -2\int_{\eta_{rec}}^{\eta} \dot{\Phi}(\eta_1)\, j_l(k(\eta - \eta_1))\, d\eta_1. \tag{32}$$

Complete solution. The complete solution to the problem is obtained by superposition:

$$\Theta_l(\eta_0) = [\Theta_0(\eta_{rec}) - \Phi(\eta_{rec})]j_l(k(\eta_0 - \eta_{rec})) + 3\Theta_1(\eta_{rec})j_l'(k(\eta_0 - \eta_{rec})) - 2\int_{\eta_{rec}}^{\eta_0} \dot{\Phi}(\eta_1)\, j_l(k(\eta_0 - \eta_1))\, d\eta_1 \tag{33}$$

for $l \geq 1$. (For $l = 0$ this gives $\bar{\Theta}_0$.) These three terms are generally called the monopole, dipole, and integrated Sachs-Wolfe (ISW) terms.

4 Large scales: the Sachs-Wolfe effect

We begin our study with the largest scales in the CMB, those that were outside the horizon at recombination. This requires $k\eta_{rec} < 1$ or

$$l < \frac{r_{CMB}}{\eta_{rec}} \approx \frac{\eta_0}{\eta_{rec}} = 50. \tag{34}$$

In this case, at recombination, we may take as an initial condition the matter-dominated photon multipole moments from the lectures on inhomogeneities:

$$\Theta_0 = \frac{3}{5}\Phi(0); \quad \Theta_1 = \frac{3}{5}i\Phi(0)(k\eta)^{1/2}; \quad \Phi = \frac{9}{10}\Phi(0). \tag{35}$$

The higher multipoles are zero at $\eta = \eta_{rec}$. Since $k\eta_{rec} \ll 1$, the dipole term in Eq. (33) is negligible. We will also ignore the effects of Λ, so that we are in matter domination and $\dot{\Phi} = 0$ (i.e. no ISW effect). Then:

$$\Theta_l(\eta_0) = -\frac{3}{10}\Phi(0)j_l(k(\eta_0 - \eta_{rec})) \approx -\frac{3}{10}\Phi(0)j_l(k\eta_0) = -\frac{1}{5}\zeta j_l(k\eta_0). \tag{36}$$

The CMB power spectrum is then:

$$C_l = \frac{4\pi}{25}\int \Delta_\zeta^2(k)|j_l(k\eta_0)|^2\,d\ln k. \tag{37}$$

Now let's suppose that $\Delta_\zeta^2(k)$ is a power law, as predicted by inflation with a smooth potential:

$$\Delta_\zeta^2(k) = \Delta_\zeta^2(\eta_0^{-1})(k\eta_0)^{n_s-1}; \tag{38}$$

then

$$\begin{aligned} C_l &= \frac{4\pi}{25}\Delta_\zeta^2(\eta_0^{-1})\int_0^\infty (k\eta_0)^{n_s-1}|j_l(k\eta_0)|^2\,d\ln k \\ &= \frac{4\pi}{25}\Delta_\zeta^2(\eta_0^{-1})\int_0^\infty x^{n_s-1}|j_l(x)|^2\frac{dx}{x}. \end{aligned} \tag{39}$$

The last integral can be evaluated to give:

$$C_l = \frac{2^{n_s-2}\pi^2}{25}\Delta_\zeta^2(\eta_0^{-1})\frac{\Gamma(l+n_s/2-1/2)\Gamma(3-n_s)}{\Gamma(l+5/2-n_s/2)\Gamma^2(2-n_s/2)}. \tag{40}$$

An important case is $n_s = 1$, for which we get:

$$C_l = \frac{2\pi}{l(l+1)}\frac{\Delta_\zeta^2}{25}. \tag{41}$$

(In this case Δ_ζ^2 is constant and doesn't need a wavenumber.) Therefore,

$$\frac{l(l+1)}{2\pi}C_l = \frac{\Delta_\zeta^2}{25} = \text{constant}. \tag{42}$$

For this reason, CMB observers often make plots of the angular power spectrum with $l(l+1)C_l/2\pi$ on the vertical axis.

If $n_s \neq 1$, but $l \gg 1$, then we can apply Stirling's formula to the Γs and get:

$$C_l \to \frac{2^{n_s-2}\pi^2\Gamma(3-n_s)}{25\Gamma^2(2-n_s/2)}\Delta_\zeta^2(\eta_0^{-1})l^{n_s-3}, \tag{43}$$

so

$$\frac{l(l+1)}{2\pi}C_l \propto l^{n_s-1}. \tag{44}$$

For $n_s > 1$ this means the CMB power spectrum increases as one goes to smaller angular scales, while for $n_s < 1$ the opposite occurs. In principle one can measure n_s this way. In practice the ISW effect is important at the lowest l's, where the Bessel function in Eq. (33) is slowly varying, and there is a limited range of l's satisfying $l < \eta_0/\eta_{rec}$. Therefore in order to measure n_s one resorts to a global fit to all the CMB data which includes scales that were inside the horizon at recombination. We will study those next.

5 Acoustic peaks

Now let's consider the scales that were inside the horizon at the time of equality.

Using Eq. (33), the anisotropy today is:

$$C_l = 4\pi \int \Delta_\zeta^2(k) \left| \frac{\bar\Theta_0}{\zeta}(k)\, j_l(k\Delta\eta) + 3\frac{\bar\Theta_1}{\zeta}(k)\, j_l'(k\Delta\eta) \right|^2 d\ln k, \tag{45}$$

where $\Delta\eta = \eta_0 - \eta_{rec}$.

We may expand this as:

$$\begin{aligned} C_l \quad = \quad & 4\pi\Bigg[\int \Delta_\zeta^2(k) \left|\frac{\bar\Theta_0}{\zeta}(k)\right|^2 [j_l(k\Delta\eta)]^2 d\ln k \\ & +9\int \Delta_\zeta^2(k) \left|\frac{\bar\Theta_1}{\zeta}(k)\right|^2 [j_l'(k\Delta\eta)]^2 d\ln k \\ & +6\int \Delta_\zeta^2(k) \Re\left(\frac{\bar\Theta_0^*}{\zeta}(k)\frac{\bar\Theta_1}{\zeta}(k)\right) j_l(k\Delta\eta) j_l'(k\Delta\eta) d\ln k\Bigg]. \end{aligned} \tag{46}$$

These correspond to the monopole perturbations that we see on the surface of last scattering, the dipole (Doppler) perturbations, and a cross-correlation term.

We can simplify the integrals if we go to late times, $k\Delta\eta \gg 1$, and use the asymptotic form of the spherical Bessel functions for $l \gg 1$. The function $j_l(x)$ goes to zero if $x < l + 1/2$, and for $x > l + 1/2$ we have:

$$\begin{aligned} j_l(x) \quad & \rightarrow \quad \frac{1}{l+1/2}\frac{\cos\beta}{\sqrt{\sin\beta}}\cos\left[\left(l+\frac{1}{2}\right)(\tan\beta-\beta)-\frac{\pi}{4}\right], \\ j_l'(x) \quad & \rightarrow \quad \frac{-1}{l+1/2}\cos\beta\sqrt{\sin\beta}\sin\left[\left(l+\frac{1}{2}\right)(\tan\beta-\beta)-\frac{\pi}{4}\right], \\ x \quad & = \quad \left(l+\frac{1}{2}\right)\sec\beta, \end{aligned} \tag{47}$$

where $0 \le \beta < \pi/2$. (These are WKB solutions and will be derived on the homework.) Now the point is that in the above integrals for C_l, we can exchange

k for β:

$$k = \frac{\sqrt{3}}{\eta_{rec}} \left(l + \frac{1}{2} \right) \sec\beta, \tag{48}$$

and

$$d \ln k = \tan\beta \, d\beta. \tag{49}$$

At large l, the arguments $(l + 1/2)(\tan\beta - \beta)$ are rapidly varying so we can replace the squares of Bessel functions with their cycle averages using $\cos^2, \sin^2 \to 1/2$, $\sin\cos \to 0$:

$$\begin{aligned} [j_l(x)]^2 &\to \frac{1}{2(l+1/2)^2} \frac{\cos^2\beta}{\sin\beta}, \\ [j_l'(x)]^2 &\to \frac{1}{2(l+1/2)^2} \cos^2\beta \sin\beta, \\ j_l(x) j_l'(x) &\to 0. \end{aligned} \tag{50}$$

The last result means that in the high-l limit, the correlation between the monopole and dipole terms vanish, which is what we'd expect since the dipole is equally likely to point toward the observer as away so it ought to add incoherently to the monopole.

In the C_l formula, we now have:

$$\begin{aligned} C_l &= 4\pi \left[\frac{1}{2} \int \Delta_\zeta^2(k) \left| \frac{\bar{\Theta}_0}{\zeta}(k) \right|^2 \frac{1}{(l+1/2)^2} \frac{\cos^2\beta}{\sin\beta} \tan\beta \, d\beta \right. \\ &\quad \left. + \frac{9}{2} \int \Delta_\zeta^2(k) \left| \frac{\bar{\Theta}_1}{\zeta}(k) \right|^2 \frac{1}{(l+1/2)^2} \cos^2\beta \sin\beta \tan\beta \, d\beta \right]. \end{aligned} \tag{51}$$

Further simplification, and noting that we usually plot $l(l+1)C_l/2\pi$:

$$\frac{l(l+1)}{2\pi} C_l = \int_0^{\pi/2} \Delta_\zeta^2(k) \left| \frac{\bar{\Theta}_0}{\zeta}(k) \right|^2 \cos\beta \, d\beta + 9 \int_0^{\pi/2} \Delta_\zeta^2(k) \left| \frac{\bar{\Theta}_1}{\zeta}(k) \right|^2 \cos\beta \sin^2\beta \, d\beta. \tag{52}$$

The physical meaning of this equation is that $\pi/2 - \beta$ is the angle between the Fourier mode $\mathbf{k}$ and the line of sight. The integration over $\cos\beta \, d\beta$ represents the averaging of such angles over the unit sphere, $\bar{\Theta}$ is the monopole, and the Doppler term has a $\sin\beta$ in amplitude ($\sin^2\beta$ in power) because only the line-of-sight component of the velocity is relevant.

Specific values. We argued earlier that for $k \gg k_{eq}$, the photon perturbations at the time of recombination were:

$$\Theta_0(\eta_{rec}) = -\zeta \cos \frac{k\eta_{rec}}{\sqrt{3}} \tag{53}$$

and

$$\Theta_1(\eta_{rec}) = -\frac{\zeta}{\sqrt{3}} \sin \frac{k\eta_{rec}}{\sqrt{3}}, \tag{54}$$

with the potentials $\Phi \to 0$. This correctly predicts that the functions $\bar{\Theta}_0/\zeta$ and $\bar{\Theta}_1/\zeta$ are oscillatory, and that this will give rise to oscillations in Eq. (52) since the integrands are dominated by $k \sim l/r_{CMB}$. Since the $\bar{\Theta}_0/\zeta$ and $\bar{\Theta}_1/\zeta$ are squared, the period of oscillation is now:

$$\Delta k = \frac{\pi\sqrt{3}}{\eta_{rec}}. \tag{55}$$

(Recall the period of $\cos^2$ or $\sin^2$ is π, not 2π.) This corresponds to oscillations in l of:

$$\Delta l = r_{CMB}\Delta k = \frac{\pi\sqrt{3}\, r_{CMB}}{\eta_{rec}} = \pi\sqrt{3}\,\frac{\eta_0}{\eta_{rec}} \approx 270 \tag{56}$$

for $\eta_0/\eta_{rec} = 50$. And this is indeed what we see.

A second prediction of this approximation, which does not come out correctly, is the amplitude of fluctuations. At large l, where the integral over β smooths out the oscillations, we predict:

$$\frac{l(l+1)}{2\pi} C_l \to \Delta^2_\zeta(k), \tag{57}$$

which is wrong: it overpredicts the CMB fluctuations. There are two major reasons for this: first, the amplitude of $\bar{\Theta}_0$ as written above is only valid if k is much, much greater than k_{eq}, which is not true of the modes relevant for CMB; and second, photons can diffuse relative to the baryons since $\dot{\tau}$ is not infinite. (We've also neglected neutrinos.) Both of these facts bring down the fluctuation power. In fact there is no range of k's in which one can simultaneously neglect diffusion and take $k \gg k_{eq}$. The first effect can only be derived by a numerical calculation, since there is no analytic solution to modes that are of order the horizon scale, and are near matter-radiation equality. It brings a factor of ~ 4 suppression in C_l. The second effect can be treated analytically, which we will do next.

6 The damping tail

At very small scales, we must consider the fact that photons have a finite mean free path. We will give two treatments of the effect: first an order-of-magnitude treatment, and then a treatment based on the Boltzmann hierarchy.

Diffusion length. We will first try to estimate the comoving distance that a photon can diffuse prior to recombination. The comoving mean free path of a photon is given by:

$$L_{\rm mfp,com} = \frac{1}{n_e \sigma_T a} = \frac{a^2}{n_{H,0}\sigma_T x_e}, \tag{58}$$

where $n_{H,0} = 2 \times 10^{-7}$ cm^{-3} is the comoving density of hydrogen atoms.

Now the conformal time between scatterings is:

$$\Delta\eta = \frac{1}{L_{\rm mfp,com}}. \tag{59}$$

The distance-squared traveled by a photon by diffusion adds incoherently after each scattering: (the number of scatterings is $\int d\eta/\Delta\eta$)

$$\Delta x^2 \sim \int L^2_{\rm mfp,com} \frac{d\eta}{\Delta\eta} = \int L_{\rm mfp,com}\, d\eta = \frac{1}{n_{H,0}\sigma_T} \int \frac{a^2}{x_e} d\eta. \tag{60}$$

If recombination were instantaneous ($x_e = 1$ until η_{rec}), and we assume matter domination so $a \propto \eta^2$, this would imply:

$$\Delta x^2 \sim \frac{a^2_{rec}\eta_{rec}}{5n_{H,0}\sigma_T} \sim \frac{(10^{-3})^2(300\,{\rm Mpc})}{5\cdot 2\times 10^{-7}\,{\rm cm}^{-3}\cdot 7\times 10^{-25}\,{\rm cm}^2}\,\frac{1\,{\rm Mpc}}{3\times 10^{24}\,{\rm cm}} \sim 140{\rm Mpc}^2, \tag{61}$$

so a photon can actually travel about 12 Mpc comoving prior to recombination. In reality the distance is a little bit larger because x_e drops as hydrogen begins to recombine, and hence there's a rise in the integrand before the surface of last scattering at $x_e \sim 0.1$.

Boltzmann equation treatment. The formal way to treat the diffusion effect is by including the Θ_2 term in the Boltzmann equation. Leaving out the potential, and neglecting the Θ_3 term which is suppressed relative to Θ_2 by another factor of $\dot\tau$, we get:

$$\begin{aligned}
\dot\Theta_0 &= -k\Theta_1 \\
\dot\Theta_1 &= \frac{1}{3}k(\Theta_0 - 2\Theta_2) + \dot\tau\left(\Theta_1 - \frac{1}{3}iv_b\right) \\
\dot\Theta_2 &= \frac{2}{5}k\Theta_1 + \frac{9}{10}\dot\tau\Theta_2.
\end{aligned} \tag{62}$$

If we neglect the baryon inertia ($R \ll 1$) so that the baryons come to the photon rest frame instantaneously then we can neglect the $\dot\tau$ term in $\dot\Theta_1$ because $v_b = -3i\Theta_1$. Then we find:

$$\frac{\partial}{\partial\eta}\begin{pmatrix}\Theta_0\\ \Theta_1\\ \Theta_2\end{pmatrix} = \begin{pmatrix} 0 & -k & 0 \\ \frac{1}{3}k & 0 & -\frac{2}{3}k \\ 0 & \frac{2}{5}k & \frac{9}{10}\dot\tau\end{pmatrix}\begin{pmatrix}\Theta_0\\ \Theta_1\\ \Theta_2\end{pmatrix}. \tag{63}$$

In practice $\dot\tau$ is varying, but on small scales one may make a WKB approximation and treat it as approximately constant over a cycle. One may then determine the dispersion relation of the acoustic waves by looking at the eigenvalues of the 3×3 matrix. The determinant is:

$$\begin{vmatrix} -\omega & -k & 0 \\ \frac{1}{3}k & -\omega & -\frac{2}{3}k \\ 0 & \frac{2}{5}k & \frac{9}{10}\dot\tau - \omega\end{vmatrix} = 0. \tag{64}$$

Expanding on the last row:

$$-\frac{4}{15}k^2\omega + \left(\omega^2 + \frac{1}{3}k^2\right)\left(\frac{9}{10}\dot{\tau} - \omega\right) = 0. \tag{65}$$

This is a cubic equation and has three roots. In the limit $|\dot{\tau}| \gg |\omega|$, the solutions are $\omega = \pm ik/\sqrt{3}, 9\dot{\tau}/10$. We want to know the leading-order corrections to the two oscillatory solutions (the exponentially decaying solution is not of interest). In this case, $|\omega| \ll |\dot{\tau}|$, so we approximate:

$$-\frac{4}{15}k^2\omega + \frac{9}{10}\dot{\tau}\left(\omega^2 + \frac{1}{3}k^2\right) = 0. \tag{66}$$

We let $\omega = \pm ik/\sqrt{3} + \epsilon$:

$$-\frac{4}{15}k^2\frac{\pm ik}{\sqrt{3}} + \frac{9}{10}\dot{\tau}\left(-\frac{1}{3}k^2 \pm \frac{2}{\sqrt{3}}ik\epsilon + \frac{1}{3}k^2\right) = 0. \tag{67}$$

The solution is:

$$\epsilon = \frac{4k^2}{27\dot{\tau}}, \tag{68}$$

so

$$\omega = \pm\frac{ik}{\sqrt{3}} + \frac{4k^2}{27\dot{\tau}}. \tag{69}$$

Since $\dot{\tau} < 0$ this means that the acoustic waves decay. The amplitude decays by a factor

$$\exp\left(-\int \frac{4k^2}{27|\dot{\tau}|}d\eta\right). \tag{70}$$

This is usually written as e^{-k^2/k_D^2}, where the damping scale is:

$$k_D^{-2} = \int \frac{4}{27|\dot{\tau}|}d\eta = \frac{4}{27}\int \frac{d\eta}{n_e\sigma_T a}. \tag{71}$$

Thus wavenumbers smaller than the photon diffusion length are wiped out.

In reality there are finite baryon inertia corrections to this equation, and also because the photons develop polarization which causes additional anisotropic scattering the factor of 4/27 should be 8/45.

7 Cosmology from the CMB power spectrum

We have seen that the CMB power spectrum is quite rich in features. It has the Sachs-Wolfe plateau at low l, then a series of acoustic peaks, and finally a damping tail. This spectrum allows us to obtain a number of cosmological observables.

Amplitude and slope. The overall normalization and tilt of the CMB power spectrum allow one to estimate Δ_ζ^2 and the possible scalar spectral index n_s.

Baryon density. The baryon density $\Omega_b h^2$ has two major imprints on the CMB power spectrum:

- Since baryons are pressureless, they decrease the sound speed and hence reduce the sound horizon. This stretches all of the acoustic peaks to higher l.

- Baryons are attracted to the potential wells of the dark matter (the $ik\Psi$ term in the baryon velocity equation), and thus the acoustic oscillation in $\Theta_0 - \Phi$ is offset: the positive extremes of $\Theta_0 - \Phi$ are larger than the negative extremes. Since the odd-numbered acoustic peaks are associated with $\Theta_0 - \Phi > 0$, they are enhanced relative to the even peaks. This effect increases if $\Omega_b h^2$ is increased.

Matter density. If the matter density $\Omega_m h^2$ is increased, then matter-radiation equality occurs earlier. This suppresses the high-l power spectrum (relative to the Sachs-Wolfe plateau) even more since the decying potential during the radiation era that drives the acoustic oscillations in our earlier calculation does not occur. Also the ISW effect, which enhances the first peak because the universe is not completely matter-dominated at recombination, is suppressed.

Distance to surface of last scattering. The peak positions in the CMB power spectrum are determined by the comoving angular diameter distance to the surface of last scattering, r. If r is increased then the peaks move to the right. Historically this was of importance in ruling out open Universe models.

A key issue here is parameter degeneracy: the situation where multiple parameters affect a feature. The slope of the CMB power spectrum is affected by both matter density and the primordial slope n_s, but not in the same way (matter density suppresses only the peak region and produces a unique suppression of the first peak). Two other degeneracies that we will encounter later are reionization and tensors, which also tilts the spectrum, but they produce unique features in the polarization that already have (reionization) or soon will (tensors) break the degeneracy.