1 Inhomogeneities

In this set of lectures, we turn our attention toward solving the evolution equations. We are interested here in the matter distribution at late times; we will consider the CMB anisotropies next.

The "real" way to solve the Boltzmann equations is with an ODE solver, and publicly available tools (CMBFAST; CAMB) can do this. For the purpose of analytical understanding though, we will make the following sweeping and over-simplifying approximations:

- Neglect the neutrino abundance: $f_\nu \to 0$.
- Leave out high multipoles: $\Theta_l = 0$ for $l \geq 2$. (Good before recombination since optically thick to photons; after recombination not good for photons but irrelevant for matter since photons don't scatter and contribute negligible gravity.)
- Initially we will also ignore the baryons, but we'll put them back in later.
- We will also leave out the cosmological constant until the end

In this case $\Psi = -\Phi$ and the Boltzmann equations for photons and CDM become:

$$\begin{aligned} \dot{\Theta}_0 + k\Theta_1 &= -\dot{\Phi}; \\ \dot{\Theta}_1 - \frac{1}{3}k\Theta_0 &= -\frac{1}{3}k\Phi; \\ \dot{\delta} + ikv &= -3\dot{\Phi}; \\ \dot{v} + aHv &= ik\Phi. \end{aligned} \tag{1}$$

The Einstein equations are:

$$\begin{aligned} k^2\Phi + 3aH(\dot{\Phi} + aH\Phi) &= 4\pi G a^2(\rho_m\delta + 4\rho_r\Theta_0) \\ k^2\Phi &= 4\pi G a^2\left[\rho_m\delta + 4\rho_r\Theta_0 + 3\frac{aH}{k}(i\rho_m v + 4\rho_r\Theta_1)\right] \end{aligned} \tag{2}$$

The key numbers to keep in mind are:

- Scale factor at equality, $a_{eq} = 3.2 \times 10^{-4}$.
- Wavenumber that enters horizon at equality, $k_{eq} \equiv a_{eq}H_{eq} = 0.015h$ Mpc^{-1}.

2 Scales and variables

We will distinguish between "large" scales where $k \ll k_{eq}$ and "small" scales where $k \gg k_{eq}$. In fact the only nontrivial dimensionless parameter in our equations is k/k_{eq}. To see this, let's define:

$$y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}. \tag{3}$$

We can transform independent variables to y by noting that:

$$\frac{dy}{d\eta} = aHy, \tag{4}$$

since $y \propto a$. The Hubble constant is (neglecting Λ, which we'll put in later):

$$H^2 = H_{eq}^2 \frac{y^{-3} + y^{-4}}{2}, \tag{5}$$

so

$$\frac{dy}{d\eta} = aHy = [a_{eq}y]\left[H_{eq}\sqrt{\frac{y^{-3}+y^{-4}}{2}}\right] y = \frac{k_{eq}}{\sqrt{2}}(y+1)^{1/2}. \tag{6}$$

Using the prime to denote y derivatives, and letting $Q = \sqrt{2}\, k/k_{eq}$, the Boltzmann equations become:

$$\begin{aligned} \Theta_0' + (y+1)^{-1/2} Q\Theta_1 &= -\Phi'; \\ \Theta_1' - \frac{1}{3}(y+1)^{-1/2} Q\Theta_0 &= -\frac{1}{3}(y+1)^{-1/2} Q\Phi; \\ \delta' + i(y+1)^{-1/2} Qv &= -3\Phi'; \\ v' + y^{-1}v &= i(y+1)^{-1/2} Q\Phi. \end{aligned} \tag{7}$$

To re-write the Einstein equations, we need:

$$\frac{aH}{k} = \frac{k_{eq}}{\sqrt{2}} \frac{(y+1)^{1/2}}{ky} = \frac{(y+1)^{1/2}}{Qy} \tag{8}$$

and

$$\frac{4\pi G a^2 \rho_r}{k^2} = \frac{4\pi G a_{eq}^2 y^2\ \rho_{eq} y^{-4}/2}{k_{eq}^2 Q^2/2} = \frac{3a_{eq}^2 y^{-2} H_{eq}^2/4}{k_{eq}^2 Q^2/2} = \frac{3}{2Q^2y^2}. \tag{9}$$

Therefore we get:

$$\begin{aligned} \Phi + 3\frac{y+1}{Q^2y}\Phi' + 3\frac{y+1}{Q^2y^2}\Phi &= \frac{3}{2Q^2y}\delta + \frac{6}{Q^2y^2}\Theta_0 \\ \Phi &= \frac{3}{2Q^2y^2}\left[y\delta + 4\Theta_0 + 3\frac{(y+1)^{1/2}}{Qy}(iyv + 4\Theta_1)\right] \end{aligned} \tag{10}$$

The solutions for $\Theta_{0,1}$, δ, v, and Φ depend only on Q and y. In general a numerical solution is required, but we can solve these equations analytically in two limits: $Q \ll 1$ and $Q \gg 1$.

3 Large scales

Here $Q \ll 1$. The perturbation goes through three regimes,

- Radiation era, superhorizon: $y \leq 1$, $aH/k \gg 1$.

- Matter era, superhorizon: $1 \leq y \leq Q^{-2}$, $aH/k > 1$.
- Matter era, subhorizon: $y \geq Q^{-2}$, $aH/k < 1$.

One can solve for the complete evolution of the perturbation variables because one can find a solution for $aH/k \gg 1$, and then during regime #2 we can switch to an approximation for general aH/k but assuming $y \gg 1$.

Superhorizon perturbations. For $aH/k \gg 1$, or $(y+1)^{1/2}/Qy \gg 1$, the perturbation equations become (recall $' = d/dy$ is of order y^{-1}, and that velocities and dipoles will be $\sim Q$ so we need to keep the full equations in this case):

$$\begin{aligned} \Theta_0' &= -\Phi'; \\ \Theta_1' &= \frac{1}{3}(y+1)^{-1/2}Q(\Theta_0 - \Phi); \\ \delta' &= -3\Phi'; \\ v' + y^{-1}v &= i(y+1)^{-1/2}Q\Phi; \\ (y+1)(y\Phi' + \Phi) &= \frac{1}{2}y\delta + 2\Theta_0. \end{aligned} \tag{11}$$

(The last line is the density Einstein equation.) In the initial conditions, $\Theta_0 = \delta/3$, and as one can see from the evolution equations this will remain true. Therefore the last equation can be re-written as:

$$(y+1)(y\Phi' + \Phi) = \frac{1}{2}y\delta + \frac{2}{3}\delta = \frac{3y+4}{6}\delta. \tag{12}$$

Our initial conditions had $\delta = \frac{3}{2}\Phi$, so:

$$\delta(y) = \delta(0) + \int \delta'\,dy = \frac{3}{2}\Phi(0) - 3\int \Phi'\,dy = \frac{3}{2}\Phi(0) - 3[\Phi(y) - \Phi(0)] = \frac{9}{2}\Phi(0) - 3\Phi(y). \tag{13}$$

Substituting this into the Φ equation:

$$(y+1)(y\Phi' + \Phi) = \frac{3}{4}(3y+4)\Phi(0) - \frac{1}{2}(3y+4)\Phi. \tag{14}$$

This is a linear first-order equation and it can be solved by the integrating factor method. Separate Φ' and Φ terms:

$$y(y+1)\Phi' + \frac{5y+6}{2}\Phi = \frac{3(3y+4)}{4}\Phi(0). \tag{15}$$

Normalize Φ':

$$\Phi' + \frac{5y+6}{2y(y+1)}\Phi = \frac{3(3y+4)}{4y(y+1)}\Phi(0). \tag{16}$$

The idea of the integrating factor method is that $(e^X\Phi)' = e^X(\Phi' + X'\Phi)$. So the left-hand side can become a total derivative if we find a function X whose derivative is $(5y+6)/2y(y+1)$. The required function is:

$$X = \int \frac{5y+6}{2y(y+1)}\,dy = \int \frac{-y + 6(y+1)}{2y(y+1)}\,dy = -\frac{1}{2}\ln(y+1) + 3y. \tag{17}$$

Then $e^X = y^3(1+y)^{-1/2}$, and:

$$(e^X\Phi)' = e^X \frac{3(3y+4)}{4y(y+1)}\Phi(0). \tag{18}$$

The solution for Φ is:

$$\Phi = e^{-X}\int e^X \frac{3(3y+4)}{4y(y+1)}\Phi(0)\,dy = \frac{3(1+y)^{1/2}\Phi(0)}{4y^3}\int \frac{y^2(3y+4)}{(y+1)^{3/2}}\,dy. \tag{19}$$

This integral can be turned into a sum of powers under the substitution $\xi = 1+y$. The solution is:

$$\Phi = \frac{\Phi(0)}{10y^3}\left[16(1+y)^{1/2} + 9y^3 + 2y^2 - 8y - 16\right]. \tag{20}$$

(The lower limit of the integral is set by requiring Φ to be finite at $y=0$.) As y becomes large:

$$\Phi \to \frac{9}{10}\Phi(0). \tag{21}$$

Thus on superhorizon scales, the potentials decrease by a factor of 9/10 in the transition from radiation to matter domination.

We would also like the densities and velocities; for $y \gg 1$ these become:

$$\delta = \frac{9}{2}\Phi(0) - 3\Phi = \frac{9}{2}\Phi(0) - \frac{27}{10}\Phi(0) = \frac{9}{5}\Phi(0), \tag{22}$$

and

$$\Theta_0 = \frac{\delta}{3} = \frac{3}{5}\Phi(0). \tag{23}$$

The velocity equation can be written as:

$$(vy)' = y(v' + y^{-1}v) = iy(y+1)^{-1/2}Q\Phi. \tag{24}$$

Therefore,

$$v = \frac{1}{y}\int iy(y+1)^{-1/2}Q\Phi\,dy = \frac{iQ}{y}\int \frac{\tilde{y}}{\sqrt{\tilde{y}+1}}\Phi(\tilde{y})\,d\tilde{y}. \tag{25}$$

This integral is complicated but at $y \gg 1$ the integral is dominated by $\tilde{y} \gg 1$. (The integration must start from 0 if $v(0)$ is to be finite.) Then $\Phi(\tilde{y}) = \frac{9}{10}\Phi(0)$ and:

$$v = \frac{iQ}{y}\frac{9}{10}\Phi(0)\int_0^y \tilde{y}^{1/2}\,d\tilde{y} = \frac{iQ}{y}\frac{9}{10}\Phi(0)\frac{2}{3}y^{3/2} = \frac{3}{5}iQ\Phi(0)\,y^{1/2}. \tag{26}$$

A similar procedure for Θ_1 gives:

$$\Theta_1 = -\frac{1}{5}Q\Phi(0)\,y^{1/2}. \tag{27}$$

These satisfy the relation $\Theta_1 = iv/3$, i.e. the matter and radiation do not move with respect to each other, which is expected on superhorizon scales.

Horizon crossing. We now wish to understand the late-time evolution for $y \gg 1$ but general k/aH. We don't need to follow the photons here because they don't scatter or affect the potentials. (We will need to consider them in order to understand the CMB.) The matter and Einstein equations are:

$$\begin{aligned} \delta' + iy^{-1/2}Qv &= -3\Phi'; \\ v' + y^{-1}v &= iy^{-1/2}Q\Phi; \\ \Phi &= \frac{3}{2Q^2y}\delta + \frac{9}{2Q^3y^{3/2}}iv. \end{aligned} \tag{28}$$

The initial conditions are that at small y,

$$\delta \to \frac{9}{5}\Phi(0); \quad v \to \frac{3}{5}iQ\Phi(0)\, y^{1/2}; \quad \Phi \to \frac{9}{10}\Phi(0). \tag{29}$$

Horizon crossing corresponds to $y \sim Q^{-2}$.

We can simplify matters by defining the new independent variable $s = Q^2y$. Then:

$$\begin{aligned} \frac{d\delta}{ds} + \frac{iv}{\sqrt{s}} &= -3\frac{d\Phi}{ds}; \\ \frac{dv}{ds} + \frac{v}{s} &= i\frac{\Phi}{\sqrt{s}}; \\ \Phi &= \frac{3\delta}{2s} + \frac{9iv}{2s^{3/2}}, \end{aligned} \tag{30}$$

with initial condition $v = \frac{3}{5}i\Phi(0)\sqrt{s}$, so the horizon entry of all the $Q \ll 1$ modes is described by the same single differential equation with the same initial condition.

We next make the change of dependent variables:

$$v = u + \frac{2}{3}is^{1/2}\Phi. \tag{31}$$

Our system of equations changes to:

$$\begin{aligned} \frac{d\delta}{ds} - \frac{2}{3}\Phi + \frac{iu}{\sqrt{s}} &= -3\frac{d\Phi}{ds}; \\ \frac{du}{ds} + \frac{u}{s} + \frac{2}{3}is^{1/2}\frac{d\Phi}{ds} &= 0; \\ \Phi &= \frac{3\delta}{2s} + \frac{9iu}{2s^{3/2}} - \frac{3}{s}\Phi. \end{aligned} \tag{32}$$

It is convenient to solve the last equation for δ,

$$\delta = \frac{2}{3}s\Phi - \frac{3iu}{s^{1/2}} + 2\Phi, \tag{33}$$

and then differentiate:

$$\frac{d\delta}{ds} = \frac{2}{3}\Phi + \left(\frac{2}{3}s + 2\right)\frac{d\Phi}{ds} - 3is^{-1/2}\left(\frac{du}{ds} - \frac{u}{2s}\right). \tag{34}$$

Substituting this into the $d\delta/ds$ equation gives:

$$\frac{2}{3}\Phi + \left(\frac{2}{3}s + 2\right)\frac{d\Phi}{ds} - 3is^{-1/2}\left(\frac{du}{ds} - \frac{u}{2s}\right) - \frac{2}{3}\Phi + \frac{iu}{\sqrt{s}} = -3\frac{d\Phi}{ds}. \tag{35}$$

(Cancel $2\Phi/3$.) This and the du/ds equation form a two-dependent-variable system of ODEs. The initial conditions are $u = 0$, $\Phi = \frac{9}{10}\Phi(0)$. But it is trivially seen that $u = 0$, $\Phi =$constant provides a solution to the equations! Therefore we can immediately write Φ and v as functions of y:

$$\Phi = \frac{9}{10}\Phi(0); \quad v = \frac{3}{5}iQ\Phi(0)\, y^{1/2}. \tag{36}$$

The density can be obtained by integration of the δ' equation:

$$\begin{aligned}
\delta(y) &= \delta(y_i) + \int_{y_i}^{y} \delta'(\tilde{y})\, d\tilde{y} \\
&= \frac{9}{5}\Phi(0) + \int_{y_i}^{y} [-3\Phi'(\tilde{y}) - i\tilde{y}^{-1/2}Qv(\tilde{y})]d\tilde{y} \\
&= \frac{9}{5}\Phi(0) + 3[\Phi(y_i) - \Phi(y)] - iQ\int_{y_i}^{y} \tilde{y}^{-1/2}\frac{3}{5}iQ\Phi(0)\tilde{y}^{1/2}\, d\tilde{y} \\
&= \frac{9}{5}\Phi(0) + \frac{3}{5}Q^2\Phi(0)y \\
&= \frac{3}{5}\Phi(0)(Q^2y + 3).
\end{aligned} \tag{37}$$

On scales well inside the horizon, the matter density becomes $\delta = \frac{3}{5}\Phi(0)Q^2y$. Note that it grows in proportion to y (or a).

4 Small scales

Now we're going to work in the opposite limit, $Q \gg 1$. In this case, there are once again three regimes:

- Radiation era, superhorizon: $y \le Q^{-1}$, $aH/k > 1$.
- Radiation era, subhorizon: $Q^{-1} \le y \le 1$, $aH/k < 1$.
- Matter era, subhorizon: $y \ge 1$, $aH/k \ll 1$.

Radiation era: potential evolution. In the radiation era, the potential is dominated by radiation, so we can first ignore the matter and later follow the DM evolution equations by treating DM as test particles.

The evolution equations become, in the $y \ll 1$ limit:

$$\begin{aligned} \Theta_0' + Q\Theta_1 &= -\Phi'; \\ \Theta_1' - \frac{1}{3}Q\Theta_0 &= -\frac{1}{3}Q\Phi; \\ \Phi &= \frac{6}{Q^2y^2}\left(\Theta_0 + \frac{3}{Qy}\Theta_1\right). \end{aligned} \tag{38}$$

It will be convenient to switch independent variables to:

$$x = \frac{Qy}{\sqrt{3}} = \frac{k\eta}{\sqrt{3}} \tag{39}$$

and change the dependent variable from Θ_1 to $V \equiv \sqrt{3}\Theta_1$. (The utility of this function will become obvious shortly.) Then:

$$\begin{aligned} \frac{d\Theta_0}{dx} + V &= -\frac{d\Phi}{dx}; \\ \frac{dV}{dx} - \Theta_0 &= -\Phi; \\ \Phi &= \frac{2}{x^2}\left(\Theta_0 + \frac{1}{x}V\right). \end{aligned} \tag{40}$$

The first two lines look like a forced harmonic oscillator with unit natural frequency, with the forcing depending on the radiation density. Note that the equations only depend on x and no other parameters.

We can solve the last equation for Θ_0 and eliminate it from the system:

$$\Theta_0 = -\frac{V}{x} + \frac{1}{2}x^2\Phi, \tag{41}$$

and substituting this into the other equations gives:

$$\begin{aligned} \frac{1}{x^2}V - \frac{1}{x}\frac{dV}{dx} + x\Phi + \frac{1}{2}x^2\frac{d\Phi}{dx} + V &= -\frac{d\Phi}{dx}; \\ \frac{dV}{dx} + \frac{V}{x} - \frac{1}{2}x^2\Phi &= -\Phi. \end{aligned} \tag{42}$$

The second equation allows us to eliminate dV/dx from the first equation:

$$\frac{1}{x^2}V - \frac{1}{x}\left(-\Phi - \frac{V}{x} + \frac{1}{2}x^2\Phi\right) + x\Phi + \frac{1}{2}x^2\frac{d\Phi}{dx} + V = -\frac{d\Phi}{dx}. \tag{43}$$

Simplifying this gives (after dividing out a factor of $1 + x^2/2$):

$$\frac{d\Phi}{dx} + \frac{1}{x}\Phi + \frac{2}{x^2}V = 0. \tag{44}$$

The next step is to turn this into a second-order differential equation for Φ. Multiplying by x^3:

$$x^3\frac{d\Phi}{dx} + x^2\Phi + 2xV = 0, \tag{45}$$

and taking the derivative:

$$x^3\frac{d^2\Phi}{dx^2} + 4x^2\frac{d\Phi}{dx} + 2x\Phi + 2x\frac{dV}{dx} + 2V = 0. \tag{46}$$

Now from the V equation, the last two terms evaluate to:

$$2x\frac{dV}{dx} + 2V = -2x\Phi + x^3\Phi. \tag{47}$$

Substituting this in:

$$x^3\frac{d^2\Phi}{dx^2} + 4x^2\frac{d\Phi}{dx} + x^3\Phi = 0. \tag{48}$$

We've seen this equation before. It is perhaps more familiar if we write it with $x^3\Phi$ as the dependent variable:

$$\frac{d^2}{dx^2}(x^3\Phi) - \frac{2}{x}\frac{d}{dx}(x^3\Phi) + x^3\Phi = 0. \tag{49}$$

This is exactly the same equation that we had in the case of the tensor perturbations during inflation, except that the independent variable is x (not $k\eta$) and the dependent variable is $x^3\Phi$. We need the solution where $x^3\Phi \to 0$ as $x \to 0$, which is:

$$x^3\Phi \propto \sin x - x\cos x. \tag{50}$$

At small x the right-hand side goes to $x^3/3$. Therefore we have the solution:

$$\Phi(x) = 3\Phi(0)\frac{\sin x - x\cos x}{x^3}. \tag{51}$$

At large times, the potential is oscillatory with amplitude decaying as $\sim 1/x^2$.

Radiation era: matter evolution. In order to understand the matter evolution we need to track the dark matter in this potential. At small y the dark matter obeys the equations

$$\begin{aligned} \delta' + iQv &= -3\Phi'; \\ v' + y^{-1}v &= iQ\Phi. \end{aligned} \tag{52}$$

We now eliminate v. Take the derivative of the first equation:

$$\delta'' + iQv' = -3\Phi'', \tag{53}$$

and then use the second equation to eliminate v':

$$\delta'' + iQ(-iQ\Phi - y^{-1}v) = -3\Phi''. \tag{54}$$

However the δ equation gives

$$iQv = -3\Phi' - \delta', \tag{55}$$

so:

$$\delta'' + Q^2\Phi - y^{-1}(-3\Phi' - \delta') = -3\Phi''. \tag{56}$$

If we switch variables to x:

$$\frac{d^2\delta}{dx^2} + 3\Phi + \frac{3}{x}\frac{d\Phi}{dx} + \frac{3}{x}\frac{d\delta}{dx} = -3\frac{d^2\Phi}{dx^2}. \tag{57}$$

Rearrange:

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\delta}{dx}\right) = -3\Phi - \frac{3}{x}\frac{d\Phi}{dx} - 3\frac{d^2\Phi}{dx^2}. \tag{58}$$

After the potential decays the right hand side goes to zero. In this case, $x\,d\delta/dx$ goes to a constant, so:

$$\frac{d\delta}{dx} = \text{const} + \frac{\text{const}}{x}, \tag{59}$$

and then:

$$\delta = \text{const} + \text{const}\ln x = \text{const}\ln(\text{const}\ x). \tag{60}$$

Numerical integration of the $\delta(x)$ equation is required to determine the constants. Dodelson finds:

$$\delta = 9.0\Phi(0)\ln(1.07x) = 9.0\Phi(0)\ln(0.62k\eta) = 9.0\Phi(0)\ln(0.62Qy). \tag{61}$$

Hu & Sugiyama followed the "correct" potential evolution (without our approximations) and get:

$$\delta = 9.6\Phi(0)\ln(0.44k\eta). \tag{62}$$

Radiation to matter transition. The potential from the radiation is decaying as $\sim x^{-2}$ or $\sim y^{-2}$, but that from the matter may be significant as one transitions to the matter era. When the matter contribution to the potential is important, y is general, and Q is large, our equations become:

$$\begin{aligned} \delta' + i(y+1)^{-1/2}Qv &= 0; \\ v' + y^{-1}v &= i(y+1)^{-1/2}Q\Phi; \\ \Phi &= \frac{3}{2Q^2y}\delta. \end{aligned} \tag{63}$$

(The $Q \gg 1$ restriction has been used to drop the velocity term in the last equation, and then the Φ' term in the first equation since Φ has a Q^{-2} in it.)

We can use the last equation to eliminate Φ, and then the second equation to eliminate v in favor of a second-order equation for δ. This gives the Meszaros equation:

$$\delta'' + \frac{2+3y}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = 0. \tag{64}$$

We need the two linearly independent solutions $D_1(y)$ and $D_2(y)$. This is a hypergeometric equation, as one can recognize by multiplication:

$$y(y+1)\delta'' + \left(1 + \frac{3}{2}y\right)\delta' - \frac{3}{2}\delta = 0. \tag{65}$$

Now we try the usual solution:

$$\delta = \sum_{r=0}^{\infty} b_r y^{c+r}, \tag{66}$$

where c is the lowest exponent. The y^{c-1} term gives:

$$c(c-1)b_0 + cb_0 = 0, \tag{67}$$

which forces $c = 0$. The y^{r-1} terms give:

$$[r(r-1)+r]b_r + \left[(r-1)(r-2) + \frac{3}{2}(r-1) - \frac{3}{2}\right] b_{r-1} = 0, \tag{68}$$

which simplifies to

$$b_r = \frac{(r-2)(r+1/2)}{r^2} b_{r-1}. \tag{69}$$

This is convenient because the hypergeometric series terminates at $r = 2$:

$$D_1(y) = b_0 \left(1 + \frac{3}{2}y\right). \tag{70}$$

We will choose the normalization:

$$D_1(y) = y + \frac{2}{3}, \tag{71}$$

which is convenient at late times.

We also need the other linearly independent solution. Since $c = 0$ was a double root the other solution will have a logarithm. (Of course, we already knew that from our solution in the radiation era.) The general way to solve for the second solution is to let $\delta = D_1 u$, and write a differential equation for u. The key is that the coefficient of u will drop out since u=constant is a solution, and we are left with only u' and u'' terms:

$$\left(y + \frac{2}{3}\right) u'' + \frac{\frac{7}{2}y^2 + 2y + \frac{2}{3}}{y(y+1)} u' = 0. \tag{72}$$

This can be re-arranged to solve for u''/u', which is $(\ln u)'$. One can rewrite this with partial fractions:

$$(\ln u)' = \frac{2}{y + \frac{2}{3}} + \frac{1}{y} - \frac{1}{2(y+1)}. \tag{73}$$

Integrating and exponentiating gives:

$$u' \propto \left(y + \frac{2}{3}\right)^{-2} y^{-1}(y+1)^{-1/2}. \tag{74}$$

Then integrate once more (the systematic way to integrate functions containing square roots is to define $\xi = \sqrt{y+1}$ to eliminate the square root and then use partial fractions), and plug into $\delta = D_1 u$:

$$D_2(y) = \left(y + \frac{2}{3}\right) \ln \frac{\sqrt{1+y}+1}{\sqrt{1+y}-1} - 2\sqrt{1+y}. \tag{75}$$

Of these solutions, the early time behavior is:

$$D_1(y) \to \frac{2}{3}; \quad D_2(y) \to \frac{2}{3} \ln \frac{4}{e^3 y} \quad (y \to 0). \tag{76}$$

The late time behavior is:

$$D_1(y) \to y; \quad D_2(y) \to \frac{8}{45} y^{-3/2} \quad (y \to \infty). \tag{77}$$

In order to determine the density at late times we need the coefficient of D_1 from matching at small y. Letting:

$$\delta(y) = C_1 D_1(y) + C_2 D_2(y), \tag{78}$$

we solve for the constants by linear solution of δ and δ':

$$C_1 = \frac{D_2' \delta - D_2 \delta'}{D_2' D_1 - D_2 D_1'}. \tag{79}$$

The denominator at early times becomes:

$$D_2' D_1 - D_2 D_1' \to -\frac{4}{9y}, \tag{80}$$

so

$$C_1 = \frac{3}{2}\delta + \frac{3}{2} y \ln \frac{4}{e^3 y} \delta'. \tag{81}$$

At small y our initial condition was

$$\delta \to 9.6\Phi(0) \ln(0.44Qy). \tag{82}$$

Plugging everything in gives:

$$C_1 = 9.6\Phi(0) \left[\frac{3}{2} \ln(0.44Qy) + \frac{3}{2} \ln \frac{4}{e^3 y}\right] = 14.4\Phi(0) \ln(0.088Q). \tag{83}$$

It follows that the late-time solution for the matter is:

$$\delta = 14.4\Phi(0) y \ln(0.088Q). \tag{84}$$

5 Transfer functions and growth functions

Transfer functions. The fact that $\delta \propto y$ at late times with a coefficient that depends on $\Phi(0)$ and Q suggests that we define the *transfer function*:

$$\delta = \frac{3}{5} Q^2 \Phi(0) T(k) y. \tag{85}$$

Our large-scale result is then $T(k) \to 1$ for $Q \ll 1$, hence the choice of normalization. On small scales, our results above suggest:

$$T(k) \to 24 Q^{-2} \ln(0.088Q) = 12 \frac{k_{eq}^2}{k^2} \ln \frac{k}{8k_{eq}}. \tag{86}$$

(Note the 12 and 8 are not exact.)

In the regime $k \sim k_{eq}$ we have to solve the equations numerically. This was done by Bardeen, Bond, Kaiser, and Szalay (BBKS 1986), who fit the transfer function with the formula:

$$T(k) = \frac{\ln(1+2.34q)}{2.34q} \left[1 + 3.89q + (16.2q)^2 + (5.47q)^3 + (6.71q)^4\right]^{-1/4}, \tag{87}$$

where they defined:

$$q = \frac{k}{\Omega_m h^2 \, \mathrm{Mpc}^{-1}} = \frac{13.7k}{k_{eq}} = 9.7Q. \tag{88}$$

The BBKS formula is useful for rough estimates but a numerical Boltzmann code (or at least the updated Eisenstein & Hu fitting formula) should be used for detailed calculations.

The density equation can then be written as:

$$\delta = \frac{3}{5} Q^2 \Phi(0) T(k) \frac{a}{a_{eq}}. \tag{89}$$

One often further simplifies this by eliminating Q^2 and a_{eq}:

$$\frac{Q^2}{a_{eq}} = \frac{2k^2}{a_{eq} k_{eq}^2} = \frac{2k^2}{a_{eq}^3 H_{eq}^2} = \frac{k^2}{\Omega_{m0} H_0^2}, \tag{90}$$

where the last equality follows from the Friedmann equation, and the fact that matter makes up half the energy density at equality. Then:

$$\delta = \frac{3k^2}{5\Omega_{m0} H_0^2} \Phi(0) T(k) a. \tag{91}$$

Growth functions. We've assumed above that the Universe remains matter-dominated after the epoch of equality. In reality, at low redshift an additional component (probably the cosmological constant Λ) becomes dominant. In this case, we must use Eq. (91) as an initial condition for the behavior of perturbations at later times. The density and velocity evolution equations for the matter on subhorizon scales are:

$$\begin{aligned} \dot{\delta} + ikv &= 0, \\ \dot{v} + aHv &= ik\Phi. \end{aligned} \tag{92}$$

It's convenient to use the scale factor a as the independent variable:

$$\begin{aligned} \frac{d\delta}{da} + \frac{ik}{a^2H}v &= 0, \\ \frac{dv}{da} + \frac{v}{a} &= \frac{ik}{a^2H}\Phi. \end{aligned} \tag{93}$$

The Poisson equation, $k^2\Phi = 4\pi G a^2 \rho_m \delta$, allows us to eliminate Φ from the second equation:

$$\frac{dv}{da} + \frac{v}{a} = \frac{4\pi i G\rho_m}{Hk}\delta = \frac{3iH\Omega_m(a)}{2k}\delta. \tag{94}$$

This can be substituted into the first equation to give a second-order equation for δ:

$$\frac{d^2\delta}{da^2} + \left(\frac{d\ln H}{da} + \frac{3}{a}\right)\frac{d\delta}{da} - \frac{3\Omega_m(a)}{2a^2}\delta = 0. \tag{95}$$

Note that k does not appear in this equation. Therefore whatever the behavior at late times is, it is local in real space.

Before Λ becomes important, $\Omega_m(a) \to 1$ and $d\ln H/da \to -3/(2a)$. In this case the equation above becomes

$$\frac{d^2\delta}{da^2} + \frac{3}{2a}\frac{d\delta}{da} - \frac{3}{2a^2}\delta = 0. \tag{96}$$

This is a dimensionally homogeneous equation with solutions $\delta \propto a$ and $\delta \propto a^{-3/2}$. One can find the solution with $\delta = a$ at early times and numerically integrate it forward to get the *growth function* $D(a)$. The matter density at late times is then given by:

$$\delta = \frac{3}{5}Q^2 a_{eq}^{-1}\Phi(0)T(k)D(a). \tag{97}$$

This is the form in which the density equation is usually written.

Since $\Phi(0) = \frac{2}{3}\zeta$, the matter power spectrum is then:

$$\Delta^2_\delta(k,a) = \left[\frac{2}{3}\frac{3k^2}{5\Omega_{m0}H_0^2}\Phi(0)T(k)D(a)\right]^2 \Delta^2_\zeta(k), \tag{98}$$

which simplifies to:

$$\Delta^2_\delta(k,a) = \frac{4}{25}T(k)^2D(a)^2\frac{k^4}{\Omega_{m0}^2H_0^4}\Delta^2_\zeta(k). \tag{99}$$

The ΛCDM model. In the Einstein-de Sitter case we could directly integrate the growth equation (Eq. 95) to get $D(a) = a$. With the cosmological constant an exact integration is no longer possible. The matter density as a function of scale factor is:

$$\Omega_m(a) = \frac{\rho_m}{\rho_{tot}} = \frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + \Omega_{\Lambda 0}}, \tag{100}$$

and:

$$\frac{d\ln H}{da} = -\frac{3}{2a}\Omega_m(a). \tag{101}$$

The growth equation then gives:

$$\frac{d^2D(a)}{da^2} + \frac{3}{2a}\left(-\frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + \Omega_{\Lambda 0}} + 2\right)\frac{dD(a)}{da} - \frac{3}{2a^2}\frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + \Omega_{\Lambda 0}}D(a) = 0. \tag{102}$$

This can be integrated; for $\Omega_{m0} = 0.3$, $\Omega_{\Lambda 0} = 0.7$, the result is that initially $D(a) = a$, but at late times $D(a) < a$ and today at $a = 1$ we have $D(1) = 0.78$. This "growth suppression" is the result of the cosmological constant causing the Universe to (i) accelerate expansion so that large a is reached before the density perturbations can grow; and (ii) dilute the dark matter $\Omega_m(a) < 1$ so that the potential wells are not as deep as they would have been in Einstein-de Sitter.

The suppression of $D(a)$ has been measured via the integrated Sachs Wolfe effect (next set of lectures). It is an important test of dark energy models because different theories produce different amounts of suppression.

6 Baryonic effects in the transfer function

Up until now we have considered the simplest model universe with no baryons and neutrinos. Of course, this is wrong and we're going to fix these problems now. We'll address the baryons first, and then discuss the neutrinos.

In the limit where $\Omega_b \ll \Omega_m$, the baryons have no effect on the transfer function $T(k)$ except to provide a source of scattering. In the real Universe, $\Omega_b < \Omega_m$, but not $\ll$. So we will first evaluate, in a very crude approximation, what the baryons do to the transfer function. Along the way we will discover a powerful new cosmological probe.

Tight coupling. Let us first consider the behavior of the baryons before recombination. If $\dot{\tau}$ is large, then the baryons are forced to move at the same velocity as the photons:

$$v_b = -3i\Theta_1. \tag{103}$$

It follows that the baryon-to-photon ratio remains fixed. Mathematically,

$$\dot{\delta}_b = -3\dot{\Phi} - ikv_b = -3\dot{\Phi} - 3ik\Theta_1 = 3\dot{\Theta}_0; \tag{104}$$

and since $\delta_b = 3\Theta_0$ initially, the baryons maintain $\delta_b = 3\Theta_0$ as long as $\dot{\tau}$ remains large, i.e. until recombination. (Except for diffusion which we'll consider later.)

Since the photons and baryons move together, and with no anisotropic stress Θ_2, we can think of them as forming a *photon-baryon fluid*.

On large scales, $k\eta_{rec} \ll 1$, all of this is of no importance because the baryon and dark matter density perturbations and velocities are the same, $\delta_b = \delta_c$ and $v_b = v_c$, until recombination. (After recombination the baryons and DM are both cold and behave similarly.) However on small scales, $k\eta_{rec} \gg 1$, the situation is very different. Let's consider a wavenumber that enters the horizon before equality, $k > k_{eq}$. (Note that $k_{eq} \approx 1.5\eta_{rec}^{-1}$, so this is true for most modes that were inside the horizon at recombination.) Then before recombination ($a < a_{eq}$) and inside the horizon, we have (from the Einstein equation):

$$\Phi = \frac{6}{Q^2y^2}\Theta_0 = \frac{6}{(k\eta)^2}\Theta_0. \tag{105}$$

We can then solve for the photon temperature perturbation Θ_0:

$$\Theta_0 = \frac{(k\eta)^2}{6}\Phi. \tag{106}$$

But for $k\eta \gg 1$, we already solved for the potential:

$$\Phi = 3\Phi(0)\frac{\sin x - x\cos x}{x^3} \to -3\Phi(0)\frac{\cos x}{x^2} = -9\Phi(0)\frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2}. \tag{107}$$

The temperature perturbation is thus:

$$\Theta_0 \to -\frac{3}{2}\cos\frac{k\eta}{\sqrt{3}}\,\Phi(0). \tag{108}$$

Since Φ is decaying at $k\eta \gg 1$, the photon dipole is:

$$\Theta_1 = -\frac{\dot{\Theta}_0}{k} = -\frac{\sqrt{3}}{2}\sin\frac{k\eta}{\sqrt{3}}\,\Phi(0). \tag{109}$$

The oscillatory behavior is not surprising since in the absence of potentials and with enough scattering to eliminate Θ_2, the photon equations are:

$$\begin{aligned} \dot{\Theta}_0 &= -k\Theta_1; \\ \dot{\Theta}_1 &= \frac{1}{3}k\Theta_0. \end{aligned} \tag{110}$$

This is the equation of a harmonic oscillator with (conformal) angular frequency $k/\sqrt{3}$. Physically it corresponds to a wave in the photon-baryon fluid. The sound speed is

$$\sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{d(\rho/3)}{d\rho}} = \frac{1}{\sqrt{3}}, \tag{111}$$

since the photons provide both the restoring force (radiation pressure) and the inertia ($\rho_\gamma \gg \rho_b$).

The above differential equations for Θ_0 and Θ_1 remain valid even into the matter-dominated era, so long as $\rho_\gamma \gg \rho_b$ (which remains valid until recombination, with minor corrections that are in the numerical codes). They are valid even at recombination. Therefore on small scales, $k \gg k_{eq}$:

$$\begin{aligned}
\delta_b(\eta_{rec}) &= -\frac{9}{2}\cos\frac{k\eta_{rec}}{\sqrt{3}}\;\Phi(0) \\
v_b(\eta_{rec}) &= \frac{3\sqrt{3}}{2}i\sin\frac{k\eta_{rec}}{\sqrt{3}}\;\Phi(0).
\end{aligned} \tag{112}$$

Post-recombination evolution. Within the horizon and after recombination, the dark matter and baryon perturbations become: (use matter domination to conclude $a \propto \eta^2$ so $aH = 2/\eta$)

$$\begin{aligned}
\dot{\delta}_c &= -ikv_c; \\
\dot{v}_c &= -\frac{2}{\eta}v_c + ik\Phi; \\
\dot{\delta}_b &= -ikv_b; \\
\dot{v}_b &= -\frac{2}{\eta}v_b + ik\Phi; \\
k^2\Phi &= 4\pi Ga^2(\rho_b\delta_b + \rho_c\delta_c).
\end{aligned} \tag{113}$$

The last equation can be re-written as: (using the Friedmann equation)

$$k^2\Phi = \frac{3}{2}a^2H^2[\Omega_b(a)\delta_b + \Omega_c(a)\delta_c] = \frac{6}{\eta^2}[\Omega_b(a)\delta_b + \Omega_c(a)\delta_c]. \tag{114}$$

Now during the matter-dominated era, $\Omega_b(a)$ and $\Omega_c(a)$ are fixed, and sum to 1:

$$\Omega_b(a) + \Omega_c(a) = 1. \tag{115}$$

In order to solve the post-recombination evolution, we will take a linear combination of the baryon and CDM equations. We define the total density and velocity:

$$\begin{aligned}
\delta_m &= \Omega_b(a)\delta_b + \Omega_c(a)\delta_c \\
v_m &= \Omega_b(a)v_b + \Omega_c(a)v_c,
\end{aligned} \tag{116}$$

and the difference density and velocity:

$$\begin{aligned}
\delta_d &= \delta_b - \delta_c \\
v_d &= v_b - v_c.
\end{aligned} \tag{117}$$

These obey the differential equations:

$$\begin{aligned}
\dot{\delta}_m &= -ikv_m \\
\dot{v}_m &= -\frac{2}{\eta}v_m + \frac{6i}{k\eta^2}\delta_m \\
\dot{\delta}_d &= -ikv_d \\
\dot{v}_d &= -\frac{2}{\eta}v_d.
\end{aligned} \tag{118}$$

We have decoupled the total matter density and velocity equations (δ_m, v_m) from the baryon-CDM difference (δ_d, v_d). The latter are not sourced by gravity, and in fact have no growing modes. This leads us to the important conclusion that even though baryons and CDM are separated at recombination, the relative density $\delta_b - \delta_c$ does not grow, and instead remains of order its value at recombination, $\sim 10^{-3}$. In studies of large scale structure and galaxy formation it is usually neglected.

Instead it is usually sufficient to follow the total matter perturbations δ_m, v_m, since by low redshifts (where galaxies form and are observable) the baryons and CDM perturbations are the same. These equations are the same as what we derived for CDM with negligible baryons, and the conversion to a second-order ODE for $\delta_m(a)$ is straightforward:

$$\frac{d^2\delta_m}{da^2} + \frac{3}{2a}\frac{d\delta_m}{da} - \frac{3}{2a^2}\delta_m = 0. \tag{119}$$

The solutions are $\delta_m \propto a, a^{-3/2}$. The general solution, starting from a_{rec}, is:

$$\delta_m = \left[\frac{3\delta_m(a_{rec})}{5a_{rec}} + \frac{2}{5}\frac{d\delta_m(a_{rec})}{da}\right] a + \left[\frac{2a_{rec}^{3/2}\delta_m(a_{rec})}{5} - \frac{2a_{rec}^{5/2}}{5}\frac{d\delta_m(a_{rec})}{da}\right] a^{-3/2}. \tag{120}$$

The first term dominates at late times. In order to compute it we need both baryon and CDM terms. In these notes I will take the CDM contribution from our solution neglecting baryons, which is not self-consistent but gives a qualitatively correct answer. (It however, is not accurate to first order in Ω_b.)

For the CDM:

$$\delta_c(a_{rec}) = \frac{3k^2}{5\Omega_{m0}H_0^2}\Phi(0)T_c(k)a_{rec}, \tag{121}$$

where $T_c(k)$ is the transfer function that we computed for CDM only. The derivative is $d\delta_c/da = \delta_c/a$ since $\delta_c \propto a$. We then get:

$$\frac{3\delta_c(a_{rec})}{5a_{rec}} + \frac{2}{5}\frac{d\delta_c(a_{rec})}{da} = \frac{3k^2}{5\Omega_{m0}H_0^2}\Phi(0)T_c(k). \tag{122}$$

For the baryons, on small scales:

$$\delta_b(a_{rec}) = -\frac{9}{2}\cos\frac{k\eta_{rec}}{\sqrt{3}}\ \Phi(0). \tag{123}$$

The derivative is:

$$\frac{d\delta_b(a_{rec})}{da} = -\frac{ik\eta}{2a_{rec}}v_b(a_{rec}) = \frac{3\sqrt{3}}{4a_{rec}}k\eta_{rec}\sin\frac{k\eta_{rec}}{\sqrt{3}}\ \Phi(0). \tag{124}$$

Since by hypothesis $k\eta_{rec} \gg 1$, the second term dominates, and we get:

$$\frac{3\delta_b(a_{rec})}{5a_{rec}} + \frac{2}{5}\frac{d\delta_b(a_{rec})}{da} = \frac{3\sqrt{3}}{10a_{rec}}k\eta_{rec}\sin\frac{k\eta_{rec}}{\sqrt{3}}\ \Phi(0). \tag{125}$$

The final result for the density at $a \gg a_{rec}$ is:

$$\delta_m(a) = \frac{3k^2}{5\Omega_{m0}H_0^2}\Phi(0)T(k)a, \tag{126}$$

where the overall transfer function is:

$$T(k) = \Omega_c(a)T_c(k) + \frac{\sqrt{3}}{2}\Omega_b(a)\frac{\Omega_{m0}H_0^2\eta_{rec}}{ka_{rec}}\sin\frac{k\eta_{rec}}{\sqrt{3}}. \tag{127}$$

We can simplify by using $\eta = 2/aH$ during matter domination and then the Friedmann equation:

$$\eta_{rec}^2 = \frac{4}{a_{rec}^2 H_{rec}^2} = \frac{4a_{rec}}{\Omega_{m0}H_0^2}. \tag{128}$$

Using this, and $\Omega_c(a) = \Omega_c/\Omega_m$, etc.:

$$T(k) = \frac{\Omega_c}{\Omega_m}T_c(k) + 2\frac{\Omega_b}{\Omega_m}\frac{\sin(k\eta_{rec}/\sqrt{3})}{k\eta_{rec}/\sqrt{3}}. \tag{129}$$

Recall that this is only good for $k\eta_{rec} > 1$; as $k \to 0$ we still have $T(k) \to 1$.

This is a rather crude form for the transfer function, but it serves to illustrate the main points. First, in models with baryons there is a suppression of the transfer function at small scales since $\Omega_c/\Omega_m < 1$. (In reality the suppression is stronger.) Secondly, there is an oscillation imprinted on the transfer function, and hence on the matter power spectrum $P(k)$, with a period:

$$\Delta k = \frac{2\pi\sqrt{3}}{\eta_{rec}} \approx 0.04\,\mathrm{Mpc}^{-1}. \tag{130}$$

The above equation incorrectly suggests that the oscilaltions go on forever; this is in fact not correct because we haven't accounted for damping of the acoustic oscillation by diffusing photons (i.e. finite $\dot{\tau}$). Also since the baryons have finite inertia, so that ρ_b/ρ_γ is significant at recombination, there is a slight slowing of the sound speed, so that Δk is actually a bit larger than the above estimate.

Baryon acoustic oscillations. These oscillations are commonly referred to as the *baryon acoustic oscillations* or BAO.

The oscillations are important because, assuming that we understand recombination, we can compute Δk in comoving units. We can then look for these features in the power spectrum of the galaxy distribution. It provides a standard ruler: observation of the angular scale, combined with knowledge of the comoving length scale Δk^{-1}, enables determination of the comoving angular diameter distance. The oscillations are a distinctive feature, not easily confused by the messy physics of galaxy formation.

To be more quantitative: let's define the correlation function of the matter,

$$\xi(x) = \langle\delta(0)\delta(x)\rangle, \tag{131}$$

which is the Fourier transform of the power spectrum:

$$\xi = \int \frac{d^3\mathbf{k}}{(2\pi)^3} P_\delta(k) e^{ikx} = \int \Delta_\delta^2(k) \frac{\sin kx}{kx} d\ln k. \tag{132}$$

This correlation function will have a bump at the BAO scale,

$$s = \frac{2\pi}{\Delta k} \approx 150\,\mathrm{Mpc\ comoving}. \tag{133}$$

Using any tracer of the matter distribution, e.g. galaxies, one can then search for an excess in pairs of galaxies at any separation and thus identify the scale s. In the transverse direction, if the excess pairs occur at an angular separation θ, we have:

$$\theta = \frac{s}{r} \quad r = \frac{s}{\theta}, \tag{134}$$

where r is the comoving angular diameter distance. The scale s must be computed by fitting to the CMB data in order to use it as a standard ruler.

If we have accurate redshifts for our galaxies, we can look for excess pairs in the radial direction as well:

$$\Delta z = \frac{s}{d\chi/dz} = sH(z). \tag{135}$$

The strength of the BAO feature depends on galaxy formation and nonlinear corrections (more on this later), but its position is remarkably robust, with simulations typically finding shifts of $\sim 1\%$. The feature was first detected in 2005 by Eisenstein et al. using 46748 elliptical galaxies in the Sloan Digital Sky Survey. They find:

$$\left[r^2 \frac{z}{H(z)}\right]^{1/3} (z = 0.35) = 1370 \pm 64\,\mathrm{Mpc}. \tag{136}$$

Future surveys will have more volume, and a higher density of galaxies. This will enable more precise determination of the distance scale. Because volume is required to get good measurements of the BAO scale, the constraints from BAO will be most useful at high redshift.

7 Neutrinos

Up to this point, we have not considered the neutrinos. But they are there (we know from BBN ^{4}He) and the neutrino oscillation data show that some of them are massive. So here we will consider, at order of magnitude level, what neutrinos do to the power spectrum.

Basic scales. Suppose for simplicity that the neutrino masses are nearly degenerate, $m_\nu \gg (\Delta m_\nu^2)^{1/2}$. The average momentum of a neutrino is:

$$\langle p \rangle = \frac{7\pi^4}{135\zeta(3)} T_\nu = 4.2 T_\nu. \tag{137}$$

The neutrino temperature redshifts as $1/a$, or more explicitly:

$$T_\nu = T_{\nu 0} a^{-1}, \quad T_{\nu 0} = 1.95\,\mathrm{K}. \tag{138}$$

The neutrinos go nonrelativistic when $\langle p \rangle \sim m_\nu$, or:

$$a_{nr} \sim \frac{4.2 T_{\nu 0}}{m_\nu} = 0.007 \left(\frac{m_\nu}{0.1\,\mathrm{eV}} \right)^{-1}. \tag{139}$$

After this time, the neutrinos start to affect the expansion of the Universe as if they were nonrelativistic matter. Their density today is:

$$\Omega_{\nu 0} = \frac{3 H_0^2}{8\pi G} n_\nu m_\nu = 0.006 \left(\frac{m_\nu}{0.1\,\mathrm{eV}} \right). \tag{140}$$

Thus the neutrinos are a minority constituent. They are also a form of dark matter in the sense that they are noninteracting. But they don't act like CDM, because they are "hot" (HDM): they have a huge velocity dispersion. The typical velocities of the neutrinos can be obtained from their momenta, which redshift as $\sim 1/a$:

$$\langle v_\nu \rangle = \frac{\langle p \rangle}{m_\nu} = \frac{a_{nr}}{a} \sim 2000 a^{-1} \left(\frac{m_\nu}{0.1\,\mathrm{eV}} \right)^{-1} \mathrm{km\,s^{-1}}. \tag{141}$$

Today the neutrinos are moving at of order 1000 km/s or larger, which is greater than the escape velocities of structures (with the possible exception of galaxy clusters). So unlike CDM the neutrinos are not collected into galaxies.

A key quantity of interest for us will be the typical comoving distance that a neutrino can travel in a Hubble time (the "free-streaming" length). During the matter-dominated era, this is:

$$L \sim \frac{\langle v_\nu \rangle}{aH} = \frac{a_{nr}}{a^2 H} = \frac{a_{nr}}{a^{1/2} \Omega_{m0}^{1/2} H_0} \sim 40 a^{-1/2} \left(\frac{m_\nu}{0.1\,\mathrm{eV}} \right)^{-1} h^{-1}\,\mathrm{Mpc}. \tag{142}$$

The key here is the negative exponent of a: the free-streaming length declines as the Universe expands. At the nonrelativistic transition, this is:

$$L_{nr} \sim 500 \left(\frac{m_\nu}{0.1\,\mathrm{eV}} \right)^{-1/2} h^{-1}\,\mathrm{Mpc}. \tag{143}$$

Growth of structure with neutrinos. Let's consider the matter evolution equations again during the matter-dominated era:

$$\begin{aligned} \dot{\delta}_m &= -ikv_m; \\ \dot{v}_m + aHv_m &= ik\Phi; \end{aligned}$$
$$\Phi = \frac{3}{2} \left(\frac{aH}{k} \right)^2 \Omega_{eff}(a) \delta_m. \tag{144}$$

Here Ω_{eff} is the fraction of the critical density in matter that can cluster. For scales of interest this always includes CDM and baryons. On scales larger than the free-streaming length $k < L^{-1}$, neutrinos cannot transport momentum and are cold: they contribute to $\Omega_{eff}(a)$ in this equation. However on scales smaller than the free-streaming length $k > L^{-1}$, the neutrinos will be smoothly distributed and hence don't contribute to the potential (but they do contribute to the mean density and hence to H^2 via the Friedmann equation). In this case they don't contribute to $\Omega_{eff}(a)$. That is:

$$\Omega_{eff}(a) = \begin{cases} 1 & k < L^{-1} \\ 1 - \Omega_{\nu 0}/\Omega_{m0} & k > L^{-1} \end{cases}. \tag{145}$$

We can solve the density equation by converting it to second-order:

$$\ddot{\delta}_m + aH\dot{\delta}_m - \frac{3}{2}(aH)^2\Omega_{eff}\delta_m = 0. \tag{146}$$

Using $aH = 2/eta$:

$$\ddot{\delta}_m + \frac{2}{\eta}\dot{\delta}_m - \frac{6}{\eta^2}\Omega_{eff}\delta_m = 0. \tag{147}$$

This is dimensionally homogeneous, so we take $\delta_m \propto \eta^c$ as a solution:

$$c(c-1) + 2c - 6\Omega_{eff} = 0, \tag{148}$$

or

$$c = \frac{-1 \pm \sqrt{1 + 24\Omega_{eff}}}{2}. \tag{149}$$

Taking the growing mode (+) and noting that $\eta \propto a^{1/2}$, we get:

$$\delta_m \propto a^{(-1+\sqrt{1+24\Omega_{eff}})/4}. \tag{150}$$

On large scales, $k < L^{-1}$, we have $\Omega_{eff} = 1$ and hence:

$$\delta_m \propto a, \tag{151}$$

as expected. On small scales, $\Omega_{eff} = 1 - \Omega_{\nu 0}/\Omega_{m0}$. If we Taylor-expand the exponent for small $\Omega_{\nu 0}$, we get:

$$\delta_m \propto a^{1-3\Omega_{\nu 0}/5\Omega_{m0}}. \tag{152}$$

In terms of the neutrino mass:

$$\frac{d\ln\delta_m}{d\ln a} = 1 - 0.013\left(\frac{m_\nu}{0.1\,\mathrm{eV}}\right). \tag{153}$$

The neutrinos thus produce a tiny suppression of the growth of structure on small scales. But this occurs in an exponent, so if the neutrino mass is 0.1 eV then for every e-fold of expansion the growth of structure is suppressed by another 1.3%. For $L_{nr}^{-1} < k < L_0^{-1}$, which is the usual regime probed by large scale structure, this suppression starts at a_{nr} (more accurately, the effect of neutrino masses begins then) and ends when $k = L^{-1}$, i.e. when

$$a = a_{nr}(kL_{nr})^2. \tag{154}$$

Thus the number of e-folds of expansion during which the modified exponent is applicable is $2\ln(kL_{nr})$, and:

$$\frac{\Delta\delta_m}{\delta_m} = -0.026\left(\frac{m_\nu}{0.1\,\mathrm{eV}}\right)\ln(kL_{nr}). \tag{155}$$

The change in the matter power spectrum is twice this:

$$\frac{\Delta P(k)}{P(k)} = -0.052\left(\frac{m_\nu}{0.1\,\mathrm{eV}}\right)\ln(kL_{nr}). \tag{156}$$

So even a small neutrino mass can have a big impact on the late-time distribution of matter. In future lectures we'll discuss large scale structure and weak lensing, the two major ways of measuring this effect on the neutrino mass.