1 Generation of perturbations in inflation

In this set of lectures we will consider how the primordial perturbations could have been produced during the inflationary epoch. We will begin with the gravitational waves, since these are simpler, and then consider the scalar perturbations.

2 Gravitaitonal waves

Tensor perturbations during inflation. If the only significant matter field during inflation is the scalar field, which has no anisotropic stress at first order in perturbation theory, then the equation describing the tensor perturbations during inflation is:

$$\ddot{E} + 2aH\dot{E} + k^2 E = 0. \tag{1}$$

We are going to solve this equation on a de Sitter background, i.e. one in which H=constant. In this case, the scale factor a is related to η via:

$$\eta = -\frac{1}{aH}, \tag{2}$$

so that:

$$\ddot{E} - \frac{2}{\eta}\dot{E} + k^2 E = 0. \tag{3}$$

(Recall that η is negative!) We can understand the classical evolution of gravitational waves by solving this equation. The solution can be taking power series solutions in η:

$$E = \sum_{r=0}^{\infty} a_r \eta^{c+r}. \tag{4}$$

The various powers of η in Eq. (3) give:

$$\begin{aligned} c(c-3)a_0 &= 0; \\ (c+1)(c-2)a_1 &= 0; \\ (c+r)(c+r-3)a_r + k^2 a_{r-2} &= 0 \ \ (r \geq 2). \end{aligned} \tag{5}$$

The first equation here forces $c = 0$ or $c = 3$. The second equation then forces $a_1 = 0$. The third equation implies the recursion:

$$a_r = \frac{-k^2}{(c+r)(c+r-3)} a_{r-2}, \tag{6}$$

which has the solution $a_r = 0$ for odd r and

$$a_r = (-1)^{r/2} k^r \frac{c!}{c-1} \frac{c+r-1}{(c+r)!} a_0 \tag{7}$$

for even r (prove by induction).

There are now two linearly independent solutions which we can normalize by setting $a_0 = 1$ ($c = 0$) or $a_0 = k^3/3$ ($c = 3$). (These normalizations are arbitrary but convenient.) The first ($c = 0$) solution is then (let $r = 2s$):

$$E = -\sum_{s=0}^{\infty}(-1)^s(2s-1)\frac{(k\eta)^{2s}}{(2s)!}. \tag{8}$$

Now inside the sum the operator $2s$ can be written as $\eta\, d/d\eta$ (since this pulls down the exponent of η), which we can pull out:

$$E = -\left(\eta\frac{d}{d\eta} - 1\right)\sum_{s=0}^{\infty}(-1)^s\frac{(k\eta)^{2s}}{(2s)!}. \tag{9}$$

The sum on the right is easily recognized as the Taylor expansion of $\cos(k\eta)$. We can then simplify to:

$$E = -\left(\eta\frac{d}{d\eta} - 1\right)\cos(k\eta) = \cos(k\eta) + k\eta\sin(k\eta). \tag{10}$$

This is one of the two linearly independent solutions. The other comes from $c = 3$:

$$E = \sum_{s=0}^{\infty}(-1)^s(2s+2)\frac{(k\eta)^{2s+3}}{(2s+3)!}. \tag{11}$$

We use the same trick of extracting $2s + 2$ as an operator:

$$E = \left(\eta\frac{d}{d\eta} - 1\right)\sum_{s=0}^{\infty}(-1)^s\frac{(k\eta)^{2s+3}}{(2s+3)!}. \tag{12}$$

If one included an $s = -1$ term the sum would be the Taylor expansion of $-\sin(k\eta)$. Without this term the sum is $k\eta - \sin(k\eta)$. Thus:

$$E = \left(\eta\frac{d}{d\eta} - 1\right)[k\eta - \sin(k\eta)] = -k\eta\cos(k\eta) + \sin(k\eta). \tag{13}$$

Therefore at the *classical* level, the gravitaitonal wave amplitude during inflation behaves as a linear combination of the two possible perturbations:

$$E(\eta) = E_C[\cos(k\eta) + k\eta\sin(k\eta)] + E_S[-k\eta\cos(k\eta) + \sin(k\eta)]. \tag{14}$$

As one goes toward the end of inflation, $\eta \to 0$, we find $E(0) = E_C$.

Semiclassical calculation. In quantum mechanics one should not think of $E(\eta)$ as a precisely defined quantity, rather it is a quantum operator. In the Heisenberg picture (our choice) it is time-dependent and satisfies Eq. (3) as an operator equation. The variables E_S and E_C (the coefficients in E) are promoted to time-independent operators. This is really a semiclassical picture since it is only valid in linear perturbation theory and we haven't quantized the background spacetime (actually we don't know how to do the latter).

We should be able to statistically predict the distribution for the final state $E(0)$ *if* we know the initial state of the system. In the absence of anything better to do we will assume that the gravitaitonal wave begins in its asymptotic ground state at early times. To understand what this means, let's write the solution for $E(\eta)$ at $-k\eta \gg 1$:

$$E(\eta) \to -k\eta[-E_C \sin(k\eta) + E_S \cos(k\eta)]. \tag{15}$$

Recalling that $\eta = -1/aH$:

$$E(\eta) \to \frac{k}{aH}[-E_C \sin(k\eta) + E_S \cos(k\eta)]. \tag{16}$$

At these early times the gravitaitonal wave is subhorizon (in causal contact) and oscillating. The E_C and E_S operators are quadratures, i.e. they are the gravitational wave amplitudes a quarter-cycle apart. They are thus analogous to the x and $p/m\omega$ operators in the quantum harmonic oscillator.

Asymptotically one can define an energy density,

$$\rho_{gw} = \frac{\omega^2}{16\pi G}\langle |h_+^2| + |h_\times^2|\rangle, \tag{17}$$

where the average must be taken over several cycles and several wavelengths to make sense, ω is the gravitational wave frequency. The strains are defined as:

$$E_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Recalling that for one of the two circular polarizations:

$$h_+ = -\sqrt{\frac{3}{8}}E; \quad h_\times = \mp i\sqrt{\frac{3}{8}}E, \tag{19}$$

and noting that the physical frequency is $\omega = k/a$ (gravitational waves propagate at the speed of light!), we can re-write the energy density as:

$$\rho_{gw} = \frac{3k^2}{64\pi G a^2}\langle |E|^2\rangle. \tag{20}$$

For a portion of the Universe with unit comoving volume, the total energy in the $\mathbf{k}, m$ gravitational wave mode is:

$$U_{gw} = \rho_{gw} a^3 V = \frac{3k^2 a}{64\pi G}\langle |E|^2\rangle = \frac{3k^4}{128\pi G a H^2}\langle |E_C^2| + |E_S^2|\rangle. \tag{21}$$

(Factor of 1/2 from the averages of $\cos^2$ and $\sin^2$.) Now in the ground state of a simple harmonic oscillator, U_{gw} should equal $\omega/2$, or $k/2a$:

$$\frac{k}{2a} = \frac{3k^4}{128\pi G a H^2}\langle |E_C^2| + |E_S^2|\rangle. \tag{22}$$

Thus:

$$\langle |E_C^2| + |E_S^2| \rangle = \frac{64\pi G H^2}{3k^3}. \quad (23)$$

The sinelike and cosinelike quadratures of a harmonic oscillator have statistically equal amplitude $\langle x^2 \rangle = \langle (p/m\omega)^2 \rangle$, so we may set $\langle |E_C^2| \rangle = \langle |E_S^2| \rangle$. Then since $E(0) = E_C$, we finally have:

$$\langle |E(\eta = 0)|^2 \rangle = \frac{32\pi G H^2}{3k^3}. \quad (24)$$

In the real Universe we do not observe just a single Fourier amplitude, rather we observe the superposition of many, which by homogeneity are independent. In order to describe this, we define the *power spectrum* of E to be the contribution to the variance of E per Fourier mode:

$$P_E(k) = \langle |E(\eta = 0)|^2 \rangle = \frac{32\pi G H^2}{3k^3}, \quad (25)$$

We will often write the power spectrum in terms of the variance per logarithmic interval in k:

$$\mathrm{Var}(E) = \int \Delta_E^2(k) \frac{dk}{k}, \quad (26)$$

or

$$\Delta_E^2(k) = \frac{4\pi k^3}{(2\pi)^3} P_E(k) = \frac{k^3 P_E(k)}{2\pi^2}. \quad (27)$$

[The density of Fourier modes for unit volume is $(2\pi)^{-3}$, hence the denominator, and the volume in **k**-space per logarithmic interval in k is $4\pi k^3$.] We find:

$$\Delta_E^2(k) = \frac{16 G H^2}{3\pi}. \quad (28)$$

This is often written in terms of the strain h, each component of which is $\sqrt{3/8}$ of E and received contribution from both $m = \pm 2$ modes. Therefore:

$$\Delta_h^2(k) = \frac{3}{4} \Delta_E^2(k) = \frac{4 G H^2}{\pi}. \quad (29)$$

Note: for tensor perturbations there is a subtlety since h is not a scalar and we have to carefully define its variance. We must define it so that for a plane gravitational wave the variance is $\langle h_+^2 + h_\times^2 \rangle / 2$. The isotropic version of this definition is:

$$\mathrm{Var}(h) = \frac{1}{4} \langle h_{ij} h_{ij} \rangle. \quad (30)$$

The result that $\Delta_h^2(k) = 4GH^2/\pi$ is one of the key results in early-universe cosmology. It tells us that the quantum fluctuations of the gravitational waves ($\omega/2$ per mode) were stretched during inflation to cosmological scales. The value of H in Eq. (29) refers to the time the fluctuations leave the horizon ($\eta \approx -1/k$) since long before then the gravitaitonal waves behave adiabatically (the number of quanta per mode does not change) and after then the waves are outside the horizon and their evolution is frozen.

If one combines Eq. (29) with the Friedmann equation for H, one gets:

$$\Delta_h^2(k) = \frac{32}{3} G^2 \rho_{inf}, \tag{31}$$

where ρ_{inf} is the energy density during inflation. Thus, if the primordial gravitational waves can be detected they directly tell us ρ_{inf}. This is one reason why these waves are highly sought-after by the CMB observers. The other – arguably more important – is that such gravitational waves, if observed with a nearly scale-invariant spectrum (i.e. Δ_h^2 varying slowly with k) would provide the best evidence that inflation really happened.

3 Scalar perturbations

We now consider the scalar perturbations generated during single field inflation. In particular, we first note that the perturbations must be adiabatic. This is because in different, causally separated parts of the Universe, there is a single slowly-rolling scalar field, and when it reaches the minimum of its potential and decays there is no way for different parts of the Universe to know that there is a large-scale inhomogeneity. These different parts of the Universe must follow along the same pressure-density curve $p(\rho)$, and will produce the same photon:neutrino:baryon:CDM ratio. (An exception occurs for multiple scalar fields, in which case the value of field ϕ_2 when ϕ_1 reaches the minimum of its potential could affect the outcome of reheating.)

Perturbations near horizon scale. The evolution equation for the scalars was found to be:

$$\nabla_\mu \nabla^\mu \phi = V'(\phi). \tag{32}$$

As an exercise you will do the perturbation theory around the background value,

$$\phi = \phi^{(0)} + \delta\phi, \tag{33}$$

and get:

$$\delta\ddot{\phi} + 2aH\delta\dot{\phi} + [k^2 + a^2 V''(\phi^{(0)})]\delta\phi = 2\ddot{\phi}^{(0)}\Psi + 2\dot{\phi}^{(0)}(\dot{\Psi} + aH\Psi). \tag{34}$$

Now of the terms in brackets,

$$\frac{a^2 V''(\phi^{(0)})}{k^2} = \left(\frac{aH}{k}\right)^2 \frac{V''}{H^2} \approx \left(\frac{aH}{k}\right)^2 \frac{3V''}{8\pi G V} = 3\left(\frac{aH}{k}\right)^2 \bar{\eta}. \tag{35}$$

Since $\bar{\eta} \ll 1$, when the perturbations are inside or near the horizon scale ($k \geq aH$) we only need to keep the k^2 term:

$$\delta\ddot{\phi} + 2aH\delta\dot{\phi} + k^2\delta\phi = 2\ddot{\phi}^{(0)}\Psi + 2\dot{\phi}^{(0)}(\dot{\Psi} + aH\Psi). \tag{36}$$

Now at horizon crossing $k \sim aH$ the right-hand side is of order $aH\Psi\dot{\phi}^{(0)}$. But at the horizon scale, $\Psi \sim \delta\rho/\rho$ and $\dot{\phi}^{(0)} \sim aV'/H$, so:

$$aH\Psi\dot{\phi}^{(0)} \sim aH\frac{\delta\rho}{\rho}\frac{aV'}{H} \sim a^2\frac{V'\delta\rho}{\rho} \sim a^2\frac{V'^2\delta\phi}{V} \sim (aH)^2\frac{V'^2\delta\phi}{GV^2} \sim (aH)^2\epsilon\delta\phi. \tag{37}$$

Since $k \sim aH$ the right-hand side is negligible compared to the left-hand side for $\epsilon \ll 1$. Thus if we are near the horizon we may drop the gravitational terms in the scalar equation:

$$\delta\ddot{\phi} + 2aH\delta\dot{\phi} + k^2\delta\phi = 0. \tag{38}$$

[Warning: this won't work far outside the horizon because of factors of k/aH.]

Quantum scalar fluctuations. Equation (38) is just like the equation we wrote down for the gravitational wave amplitude, so it has the same solution:

$$\delta\phi(\eta) = \phi_C[\cos(k\eta) + k\eta\sin(k\eta)] + \phi_S[-k\eta\cos(k\eta) + \sin(k\eta)]. \tag{39}$$

At early times, the fluctuations in the scalar field have an energy density:

$$\rho_{\delta\phi} = \omega^2\langle|\delta\phi^2|\rangle. \tag{40}$$

(The kinetic energy density is $\omega^2\delta\phi^2/2$, and the gradient energy density is the same.) The scalar field fluctuations propagate at the speed of light (no mass term) so $\omega = k/a$:

$$\rho_{\delta\phi} = \frac{k^2}{a^2}\langle|\delta\phi^2|\rangle. \tag{41}$$

The energy in unit comoving volume is:

$$U_{\delta\phi} = k^2a\langle|\delta\phi^2|\rangle \to \frac{k^4}{aH^2}\langle|\phi_C^2| + |\phi_S^2|\rangle. \tag{42}$$

Just as in the gravitational wave case, we set this equal to $\omega/2 = k/2a$, and take one of the quadratures:

$$\langle|\phi_C^2|\rangle = \frac{H^2}{2k^3}. \tag{43}$$

After horizon exit, i.e. $k|\eta| < 1$, $\phi \to \phi_C$ and so the scalar field power spectrum is:

$$P_{\delta\phi}(k) = \frac{H^2}{2k^3}, \tag{44}$$

and the variance per $\ln k$ is obtained by multiplying by $k^3/2\pi^2$:

$$\Delta^2_{\delta\phi}(k) = \frac{H^2}{4\pi^2}. \tag{45}$$

However, unlike for the gravitational wave case, we're not done because the scalar field will evolve outside the horizon. In particular, it will decay, and since this happens outside the horizon we need the full machinery of GR to understand what this implies about the primordial potential.

The curvature perturbation. In order to follow the evolution through the later stages of inflation, reheating, and the early radiation era, we need a new variable ζ. This is called the *curvature perturbation* and is defined by:

$$\zeta \equiv \frac{aHj}{ik(\bar{\rho}+\bar{p})} + \Phi. \tag{46}$$

To understand the importance of the curvature perturbation, we will need the continuity equation $T^{\mu\nu}{}_{;\nu} = 0$; the $\mu = 0$ component gives:

$$\delta\dot{\rho} = -3aH(\delta\rho + \delta p) - 3(\bar{\rho}+\bar{p})\dot{\Phi} - ikj. \tag{47}$$

From the Einstein equations:

$$\begin{aligned} k^2\Phi + 3aH(\dot{\Phi} - aH\Psi) &= 4\pi G a^2 \delta\rho \\ \dot{\Phi} - aH\Psi &= 4\pi G a^2 j/ik, \end{aligned} \tag{48}$$

on large scales ($k/aH \ll 1$) we may drop the k^2 term in the first equation and get:

$$\delta\rho = 3aH\frac{j}{ik}. \tag{49}$$

Then:

$$\zeta \equiv \frac{\delta\rho}{3(\bar{\rho}+\bar{p})} + \Phi, \tag{50}$$

and in the density equation we have:

$$-ikj = \frac{k^2\delta\rho}{3aH}. \tag{51}$$

and we can substitute

$$\delta\dot{\rho} = -3aH(\delta\rho + \delta p) - 3(\bar{\rho}+\bar{p})\dot{\zeta} + (\bar{\rho}+\bar{p})\partial_\eta\left(\frac{\delta\rho}{\bar{\rho}+\bar{p}}\right) + \frac{k^2\delta\rho}{3aH}. \tag{52}$$

Now the last term is negligible for $k/aH \ll 1$. Also the derivative ∂_η acting on $\delta\rho$ cancels the left-hand side, so:

$$0 = -3aH(\delta\rho + \delta p) - 3(\bar{\rho}+\bar{p})\dot{\zeta} + (\bar{\rho}+\bar{p})\partial_\eta\left(\frac{1}{\bar{\rho}+\bar{p}}\right)\,\delta\rho. \tag{53}$$

Solving for $\dot{\zeta}$ gives:

$$\dot{\zeta} = aH\frac{\delta\rho+\delta p}{\bar{\rho}+\bar{p}} + \frac{d(\bar{\rho}+\bar{p})/d\eta}{3(\bar{\rho}+\bar{p})^2}\,\delta\rho. \tag{54}$$

But we know from continuity that:

$$aH = -\frac{d\bar{\rho}/d\eta}{3(\bar{\rho}+\bar{p})}, \tag{55}$$

so:

$$\dot{\zeta} = \frac{1}{3(\bar{\rho}+\bar{p})^2}\left[\frac{d\bar{p}}{d\eta}\delta\rho - \frac{d\bar{\rho}}{d\eta}\delta p\right]. \tag{56}$$

But we argued earlier that the perturbations must be adiabatic. Therefore δp and $\delta\rho$ are in the ratio:

$$\delta p : \delta\rho = \frac{d\bar{p}}{d\eta} : \frac{d\bar{\rho}}{d\eta}, \tag{57}$$

which means that $\dot{\zeta} = 0$. Therefore ζ *is conserved outside the horizon* (for adiabatic perturbations only).

The value of the curvature perturbation. Now we need the relation between $\delta\phi$ exiting the horizon and the curvature perturbation. The stress-energy tensor for a scalar field was:

$$[T_\phi]_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\nabla_\alpha\phi)(\nabla_\beta\phi) - g_{\mu\nu}V(\phi). \tag{58}$$

In the Newtonian gauge, $g_{\mu\nu}$ is diagonal, and the momentum density is

$$j_{\hat{i}} = -a^2\dot{\phi}\partial_i\phi. \tag{59}$$

In Fourier modes,

$$j = -ika^2\dot{\phi}^{(0)}\delta\phi. \tag{60}$$

The curvature perturbation is then: (recall $\Phi \to 0$ at the horizon, and for a scalar $\bar{\rho}+\bar{p} = a^2\dot{\phi}^{(0)2}$)

$$\zeta = \frac{aHj}{ik(\bar{\rho}+\bar{p})} = \frac{-a^3H\dot{\phi}^{(0)}\delta\phi}{a^2\dot{\phi}^{(0)2}} = \frac{-aH\delta\phi}{\dot{\phi}^{(0)}}. \tag{61}$$

This means that the primordial curvature perturbation power spectrum is:

$$\Delta^2_\zeta(k) = \left(\frac{-aH}{\dot{\phi}^{(0)}}\right)^2\Delta^2_{\delta\phi}(k) = \left(\frac{-aH}{\dot{\phi}^{(0)}}\right)^2\frac{H^2}{4\pi^2}. \tag{62}$$

Recall that

$$\dot{\phi}^{(0)} = -\frac{aV'}{H}, \tag{63}$$

so:

$$\frac{-aH}{\dot{\phi}^{(0)}} = \frac{H^2}{V'} = \frac{8\pi GV}{V'} = \sqrt{\frac{4\pi G}{\epsilon}}. \tag{64}$$

The final result for the curvature perturbation power spectrum is:

$$\Delta^2_\zeta(k) = \frac{GH^2}{\pi\epsilon}. \tag{65}$$

Initial conditions during radiation epoch. The curvature perturbation relates to the potential perturbation during radiation domination as follows. We know that:

$$\zeta = \frac{aHj}{ik(\bar{\rho}+\bar{p})} + \Phi = \frac{aH(4/3)\bar{\rho}_r v_b}{(4/3)ik\bar{\rho}_r} + \Phi = \frac{aHv_b}{ik} + \Phi = \frac{aH(ik\Phi/2aH)}{ik} + \Phi = \frac{3}{2}\Phi. \tag{66}$$

Thus the primordial gravitational potential perturbations are $\Phi = \frac{2}{3}\zeta$, and have power spectrum:

$$\Delta_\Phi^2(k) = \frac{4GH^2}{9\pi\epsilon}. \tag{67}$$

4 Primordial fluctuations from specific models

Properties of primordial fluctuations. We will now review the predictions of specific inflationary models for the primordial fluctuations. We will first introduce some notation. The *scalar spectral index* n_s is given by:

$$n_s \equiv 1 + \frac{d\ln\Delta_\zeta^2(k)}{d\ln k}. \tag{68}$$

(The 1 is convention.) The *running index* α_s is:

$$\alpha_s \equiv \frac{d^2\ln\Delta_\zeta^2(k)}{(d\ln k)^2}. \tag{69}$$

If the initial conditions are almost scale-invariant they can be expanded as a power series,

$$\ln\Delta_\zeta^2(k) = \ln\Delta_\zeta^2(k_\star) + [n_s(k_\star) - 1]\ln\frac{k}{k_\star} + \frac{1}{2}\alpha_s(k_\star)\left(\ln\frac{k}{k_\star}\right)^2 + ... \tag{70}$$

Observers typically fit the first two or three terms to their data and quote constraints. Note that in addition to spectral indices we need an absolute normalization $\Delta_\zeta(k_\star)$ at some wavenumber.

For gravitational waves, the *tensor-to-scalar ratio* r is defined by:

$$r \equiv \frac{4\Delta_h^2(k)}{\Delta_\zeta^2(k)}. \tag{71}$$

(This r is consistent with WMAP convention and most papers but not with any books.) One also defines a *tensor spectral index* n_t:

$$n_t \equiv \frac{d\ln\Delta_h^2(k)}{d\ln k}. \tag{72}$$

As an example, the recent parameters from WMAP (CMB) + BAO + SNe are: (for $k_\star = 0.002$ Mpc^{-1}):

- $\Delta_\zeta^2(k_\star) = (2.46 \pm 0.09) \times 10^{-9}$.
- $n_s = 0.960^{+0.014}_{-0.013}$ (assuming no running, tensors); 0.968 ± 0.015 (with tensors).
- $\alpha_s = -0.032^{+0.021}_{-0.020}$ (no tensors).

- $r < 0.20$ (95%CL, no running); < 0.54 (with running).

Since we haven't seen the gravitational waves yet we can't measure n_t.

Relation to slow-roll parameters. From the calculated scalar power spectrum, we can find the spectral index n_s by differentiation:

$$n_s = 1 + \frac{d \ln \Delta_\zeta^2(k)}{d \ln k} = 1 - \frac{d \ln(GH^2/\pi\epsilon)}{dN}. \tag{73}$$

But:

$$\frac{GH^2}{\pi\epsilon} \propto \frac{V^3}{V'^2}, \tag{74}$$

so the logarithmic derivative is:

$$n_s = 1 - \frac{d\phi}{dN}\left(3\frac{d\ln V}{d\phi} - 2\frac{d\ln V'}{d\phi}\right) = 1 - \frac{V'}{8\pi GV}\left(\frac{3V'}{V} - \frac{2V''}{V'}\right) = 1 - \frac{3V'^2}{8\pi GV^2} + \frac{V''}{4\pi GV} = 1 - 6\epsilon + 2\bar{\eta}. \tag{75}$$

The tensor-to-scalar ratio is:

$$r = \frac{4\Delta_h^2}{\Delta_\zeta^2} = 16\epsilon. \tag{76}$$

Thus n_s and r map directly into the slow-roll parameters. If expressed as a function of N, we may differentiate to get $\alpha_s = -dn_s/dN$.

Consistency relation. For single scalar field inflation, there is an elegant relation between r and n_t. If we recall that $\Delta_h^2 \propto H^2 \propto V$, then:

$$n_t = \frac{d\ln V}{d\ln k} = -\frac{d\ln V}{dN} = -\frac{d\phi}{dN}\frac{V'}{V} = -\frac{V'}{8\pi GV}\frac{V'}{V} = -2\epsilon. \tag{77}$$

Therefore:

$$n_t = -\frac{r}{8}. \tag{78}$$

This relation, if verified, would be a triumph, but foregrounds may turn out to be too serious of a problem.

Predictions from specific models.

Massive scalar. We have $\epsilon = \bar{\eta} = 1/2N$, so:

$$n_s = 1 - \frac{2}{N}; \quad \alpha_s = -\frac{2}{N^2}; \quad r = \frac{8}{N}. \tag{79}$$

The mass of the scalar can always be adjusted to match $\Delta_\zeta^2(k_\star)$; this requires $m \sim 2 \times 10^{13}$ GeV. If this model is correct then since we expect $N \sim 60$, $r \sim 0.13$, which means the tensors should be detectable in the next few years.

Quartic potential. We have $\epsilon = 1/N$, $\bar{\eta} = 3/2N$, so:

$$n_s = 1 - \frac{3}{N}; \quad \alpha_s = -\frac{3}{N^2}; \quad r = \frac{16}{N}. \tag{80}$$

This is ruled out by WMAP.

Inverted quartic. Here we had (for $\phi_{\rm end} = qV_0^{1/2}/\mu$, where $q \sim 1$):

$$\bar{\eta} = -\frac{\mu^2}{8\pi G V_0}, \quad \epsilon = q^2 \frac{|\bar{\eta}|}{2} e^{-2|\bar{\eta}|N}. \tag{81}$$

Then:

$$n_s = 1 - \left(2 + 3q^2 e^{-2|\bar{\eta}|N}\right)|\bar{\eta}|; \quad \alpha_s = -6q^2|\bar{\eta}|^2 e^{-2|\bar{\eta}|N}; \quad r = 8q^2|\bar{\eta}|e^{-2|\bar{\eta}|N}. \tag{82}$$

This is actually a two-parameter potential, and for any $|\bar{\eta}|$ one can choose V_0 to give the right normalization. In these models, there is an upper limit $r \leq 1.5q^2/N \sim 0.02$, so seeing the tensors will be hard (or maybe impossible).

Hybrid inflation. Here we had:

$$\bar{\eta} = \frac{m^2}{8\pi G V_0}, \quad \epsilon = q^2 \frac{\bar{\eta}}{2} e^{2\bar{\eta}N}, \tag{83}$$

where the phase transition occurred at

$$\phi_c = \frac{qV_0^{1/2}}{m}. \tag{84}$$

In this case we have to have $q \ll 1$ in order for our approximation $m^2\phi^2 \ll V_0$ to be valid. In fact since $\phi \propto e^{\bar{\eta}N}$, if the observable scales left the horizon when $m^2\phi^2 \ll V_0$, then we need $qe^{\bar{\eta}N} \ll 1$.

$$n_s = 1 + \left(2 - 3q^2 e^{2\bar{\eta}N}\right)\bar{\eta}; \quad \alpha_s = 6q^2\bar{\eta}^2 e^{2\bar{\eta}N}; \quad r = 8q^2\bar{\eta}e^{-2\bar{\eta}N}. \tag{85}$$

The tensors will only be detectable if $qe^{\bar{\eta}N}$ is only a factor of a few less than 1. There is no special reason for this to be the case, and indeed if you want the scalar field $\phi \ll M_{\rm Pl}$ (as many supergravity theorists do) then $r \ll 1$. Note that another prediction of this limiting case is that $n_s > 1$, which is either marginal or ruled out by WMAP.

But be warned that one could have a model where $qe^{\bar{\eta}N} \sim 1$, where when CMB scales left the horizon $m^2\phi^2 \sim V_0$. In this case one could have $n_s < 1$ and significant tensors, at the expense of fine tuning. Such models may also give significant $\alpha_s > 0$, and may even switch from $n_s < 1$ to $n_s > 1$.