1 Initial conditions

Now that we have the Einstein and Boltzmann equations, we need to find the appropriate initial conditions. That is the purpose of this lecture. We will do the vectors first, because they are the easiest; then the tensors; and finally the scalars.

Vector perturbations. We need the initial values of the photon and neutrino perturbations $\{\Theta_l\}_{l=1}^{\infty}$ and $\{\mathcal{N}_l\}_{l=1}^{\infty}$, and the matter velocities v_b and v_c. Let's consider the photons first; their equations are:

$$\begin{aligned}
\dot{\Theta}_1 &= -\frac{1}{\sqrt{3}}k\Theta_2 + \dot{\tau}\left(\Theta_1 - \frac{i}{3}v_b\right); \\
\dot{\Theta}_2 &= k\frac{\sqrt{3}\Theta_1 - 2\sqrt{2}\Theta_3}{5} + \frac{iB}{5\sqrt{3}} + \frac{9}{10}\dot{\tau}\Theta_2; \\
\dot{\Theta}_2 &= k\frac{\sqrt{(l-1)(l+1)}\Theta_{l-1} - \sqrt{l(l+2)}\Theta_{l+1}}{2l+1} + \dot{\tau}\Theta_l.
\end{aligned} \tag{1}$$

At small η, $\dot{\tau} \to -\infty$, so in order for the Θs to not blow up, the quantities multiplying $\dot{\tau}$ must go to zero. This means:

$$\Theta_1 \to \frac{i}{3}v_b; \quad \Theta_l \to 0 \ \ (l \geq 2). \tag{2}$$

The photons are forced to have the same velocity as the baryons due to the high opacity, and all higher moments (e.g. photon quadrupole) are wiped out.

At first glance it looks like one has more freedom in setting the initial conditions for the neutrinos, since there are no $\dot{\tau}$s in their equations. However at very early times ($t < 1$ second) the Universe is opaque to neutrinos, and so the arguments above should also apply to the neutrinos:

$$\mathcal{N}_1 \to \frac{i}{3}v_b; \quad \mathcal{N}_l \to 0 \ \ (l \geq 2). \tag{3}$$

In order to proceed we need an initial condition for thhe baryon velocity. The equation we derived in the last lecture is:

$$\dot{v}_b = -aHv_b + \frac{\dot{\tau}}{R}(v_b + 3i\Theta_1), \tag{4}$$

where again $R = 3\bar{\rho}_b/4\bar{\rho}_\gamma$. We can add this to the equation for the photons:

$$\frac{3}{R}i\dot{\Theta}_1 = -\frac{\sqrt{3}}{R}ik\Theta_2 + \frac{\dot{\tau}}{R}\left(v_b + 3i\Theta_1\right) \tag{5}$$

Subtracting these two equations gives:

$$\dot{v}_b - \frac{3}{R}i\dot{\Theta}_1 = -aHv_b - \frac{\sqrt{3}}{R}ik\Theta_2. \tag{6}$$

At early times we found $\Theta_2 \to 0$, and $\Theta_1 \to \frac{i}{3} v_b$. Substituting these in gives:

$$\dot{v}_b + \frac{1}{R} \dot{v}_b = -aHv_b, \tag{7}$$

which then implies:

$$\dot{v}_b = -aH\frac{R}{R+1} v_b. \tag{8}$$

Since $R \propto a$, if one extrapolates backward to $a = 0$, v_b approaches a constant.

The CDM equation is $\dot{v}_c = -aHv_c$ so $v_c \propto a^{-1}$; a short time after the Big Bang one should thus have $v_c \approx 0$.

It thus appears that we have one potentially nonzero initial vector mode, that with $v_b \neq 0$ and $\Theta_1 = \mathcal{N}_1 = iv_b/3$. However we still have to put this into the Einstein equation to see whether it works. We get:

$$\begin{aligned} B &= \frac{16\pi G a^2}{k^2}(\bar{\rho}_b v_b + \bar{\rho}_c v_c - 4i\bar{\rho}_\gamma \Theta_1 - 4i\bar{\rho}_\nu \mathcal{N}_1 \\ &\to \frac{16\pi G a^2}{k^2}(\bar{\rho}_\gamma + \bar{\nu})\frac{4}{3} v_b, \end{aligned} \tag{9}$$

where in the second line we have used the fact that $\bar{\rho}_b \ll \bar{\rho}_\gamma$. Using the Friedmann equations, we further find:

$$B \to 8\left(\frac{aH}{k}\right)^2 v_b. \tag{10}$$

Now at early times, $H \propto a^{-2}$ so $(aH/k)^2 \propto a^{-2}$. Therefore if v_b did approach a nonzero constant at early times the metric perturbations become very large. The early Universe in this case looks very different from FRW and this is not a well-behaved initial perturbation.

We technically need to show that the large metric perturbations are not a gauge artifact and that they blow up in any gauge. We can show this by replacing the left-hand side of Eq. (10) with the gauge-invariant combination:

$$B_i - \frac{2i}{k}\dot{E}_{i3}. \tag{11}$$

If B_i is to remain small then $\dot{E}_{i3}$ must blow up.

Therefore *there are no well-behaved primordial vector perturbations.* They are of interest if (i) vector perturbations are sourced by cosmic strings; or (ii) in collapsing universe models where the ill-behaved modes are part of the initial conditions and often prevent the universe from collapsing to an FRW singularity.

Tensor perturbations. The same arguments about $\dot{\tau}$ apply here and force all photon and neutrino moments to zero at early times. This leaves one with:

$$\ddot{E} + 2aH\dot{E} + k^2 E = 0. \tag{12}$$

At early times, we have $a \propto \eta$ so $aH = 1/\eta$:

$$\ddot{E} + \frac{2}{\eta}\dot{E} + k^2 E = 0. \tag{13}$$

If we take a trial solution of the form $E \propto \eta^c$, then these terms are of order η^{c-2}, η^{c-2}, and η^c respectively; so we may drop the last one. Equating coefficients of $c-2$ gives

$$c(c-1) + 2c = 0, \tag{14}$$

which has the solutions $c = 0$ and $c = -1$. The second solution is a decaying mode, and the former solution (E =constant) is the only one that survives. At late times (when $k\eta \geq 1$) it switches to oscillating behavior, i.e. it becomes a standing gravitational wave.

Scalar perturbations. The $\dot{\tau}$ arguments force

$$\Theta_1, \mathcal{N}_1 \to \frac{i}{3} v_b; \quad \Theta_l, \mathcal{N}_l \to 0 \ \ (l \geq 2) \tag{15}$$

just as for the vectors. This immediately implies:

$$\Psi = -\Phi. \tag{16}$$

Since we take $k\eta \ll 1$, the k-terms in the Boltzmann hierarchy go away, and we are left with

$$\dot{\Theta}_0 = \dot{\mathcal{N}}_0 = -\dot{\Phi}. \tag{17}$$

Now consider the metric perturbations. The Einstein density equation gives

$$k^2\Phi + 3aH(\dot{\Phi} - aH\Psi) = 4\pi G a^2 \delta\rho, \tag{18}$$

we derive:

$$3aH(\dot{\Phi} - aH\Psi) = 16\pi G a^2 (\bar{\rho}_\gamma \Theta_0 + \bar{\rho}_\nu \mathcal{N}_0). \tag{19}$$

(Ignore k^2 in comparison to $aH\partial_\eta$ and keep only radiation terms on the right.) But $aH = 1/\eta$ in the radiation era, and the photon and neutrino densities are:

$$\bar{\rho}_\gamma = \frac{3H^2}{8\pi G}(1 - f_\nu); \quad \bar{\rho}_\nu = \frac{3H^2}{8\pi G} f_\nu. \tag{20}$$

This results in a simplification to:

$$\frac{3}{\eta}\left(\dot{\Phi} - \frac{\Psi}{\eta}\right) = \frac{6}{\eta^2}[(1 - f_\nu)\Theta_0 + f_\nu \mathcal{N}_0]. \tag{21}$$

Simplify:

$$\eta\dot{\Phi} - \Psi = 2[(1 - f_\nu)\Theta_0 + f_\nu \mathcal{N}_0]. \tag{22}$$

Let's take the (conformal) time derivative:

$$\eta\ddot{\Phi} + \dot{\Phi} - \dot{\Psi} = 2[(1 - f_\nu)\dot{\Theta}_0 + f_\nu \dot{\mathcal{N}}_0] = -2\dot{\Phi}. \tag{23}$$

Using equality of potentials ($\Psi = -\Phi$), we can write this as a differential equation for Φ:

$$\eta\ddot{\Phi} + 4\dot{\Phi} = 0. \tag{24}$$

We can try a power law solution for Φ since this is a dimensionally homogeneous equation, the solutions are Φ =constant and $\Phi \sim \eta^{-3}$. The latter solution is decaying so only the Φ =constant piece is a viable initial condition.

It follows that Θ_0 and $\mathcal{N}_0$ go to a constant as well. From Eq. (22) we then derive:

$$\Phi = 2[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0]. \tag{25}$$

But at some early epoch the neutrinos thermalized with the photons (the possible exception is a large lepton asymmetry). In this case $\mathcal{N}_0 = \Theta_0$ at early times, and $\Phi = 2\Theta_0$.

We will also need the initial conditions for the matter densities and velocities. For the dark matter the velocity equation is:

$$\dot{v}_c = -aHv_c - ik\Psi = -aHv_c + ik\Phi. \tag{26}$$

We can simplify this by writing:

$$\frac{\partial}{\partial\eta}(av_c) = \dot{a}v_c + a\dot{v}_c = a(aHv_c + \dot{v}_c) = ika\Phi. \tag{27}$$

But $av_c \to 0$ as $a \to 0$ for a well-behaved velocity, thus:

$$v_c = \frac{1}{a}\int ika\Phi \, d\eta = \frac{ik\eta\Phi}{2}, \tag{28}$$

since $a \propto \eta$. Writing in terms of $\eta = 1/(aH)$ we find:

$$v_c = \frac{ik\Phi}{2aH}. \tag{29}$$

For the baryons the situation is more complicated because of the scattering with photons. The baryon and photon equations are:

$$\begin{aligned} \dot{v}_b &= -aHv_b + \frac{\dot{\tau}}{R}(v_b + 3i\Theta_1) - ik\Psi, \\ \dot{\Theta}_1 &= k\frac{\Theta_0 - 2\Theta_2}{3} + \dot{\tau}\left(\Theta_1 - \frac{1}{3}iv_b\right) + \frac{1}{3}k\Psi. \end{aligned} \tag{30}$$

We can take the linear combination of these that eliminates $\dot{\tau}$:

$$\dot{v}_b - \frac{3}{R}i\dot{\Theta}_1 = -aHv_b - ik\Psi - \frac{3}{R}ik\frac{\Theta_0 - 2\Theta_2}{3} - \frac{1}{R}ik\Psi. \tag{31}$$

Eliminating $\Theta_2 = 0$, and substituting $\Theta_1 = iv_b/3$ on the left side, we simplify this to:

$$\left(1 + \frac{1}{R}\right)\dot{v}_b = -aHv_b + \left(1 + \frac{1}{R}\right)ik\Phi - \frac{1}{R}ik\Theta_0. \tag{32}$$

As $R \to 0$:

$$\dot{v}_b = ik(\Phi - \Theta_0) = ik\Theta_0 = \frac{ik\Phi}{2}. \tag{33}$$

This integrates to give:

$$v_b = v_b0 + \frac{ik\Phi}{2aH}. \tag{34}$$

It looks at this point like one can have any initial velocity one wants, but just as in the situation with the vectors the Einstein equations come in here and restrict the reasonable initial conditions. Recall the momentum Einstein equation,

$$\dot{\Phi} - aH\Psi = 4\pi G a^2 \frac{j}{ik}. \tag{35}$$

At early times the left-hand side goes to Φ/η, and the momentum density goes to

$$j \to -4i\bar{\rho}_r\Theta_1 = \frac{4}{3}\bar{\rho}_r v_b. \tag{36}$$

Therefore:

$$\frac{\Phi}{\eta} = \frac{16\pi G a^2}{3}\bar{\rho}_r \frac{v_b}{ik} = \frac{2}{\eta^2}\frac{v_b}{ik}. \tag{37}$$

As $\eta \to 0$, this forces $v_b \to 0$. Thus for well-behaved initial conditions we need $v_b \to 0$.

It is easily seen from the density evolution equations that δ_b and δ_c can start out with any value.

To summarize: the initial conditions for a scalar mode are controlled by the values of Θ_0, δ_b, and δ_c. Given these, one has $\Phi = -\Psi = 2\Theta_0$, and

$$\Theta_1 = \mathcal{N}_1 = \frac{iv_b}{3} = \frac{iv_c}{3} = -\frac{k\Phi}{6aH}. \tag{38}$$

Adiabatic vs. isocurvature perturbations. We have seen that there are three allowed types of initial scalar perturbations, and we need a classification for them. We first note that in general, an observer even at early times (when the perturbation is outside the horizon, $k\eta \ll 1$) sees something different depending on whether he is in a "crest" or a "trough" of a perturbation. For example, the number density of photons is proportional to $T_\gamma^3 \propto 1 + 3\Theta_0$. Therefore the photon-to-baryon ratio, which is constant with time for a given observer, is modulated by $3\Theta_0 - \delta_b$. So if $3\Theta_0 - \delta_b \neq 0$, then even at early times observers in different regions see different phenomenology.

These comparisons become impossible in models in which all observers agree on the basic quantities like the photon:baryon ratio, i.e. if

$$\Theta_0 : \delta_b : \delta_c = 3 : 1 : 1. \tag{39}$$

Such an initial perturbation is called *adiabatic*.

On the other hand, one could have a perturbation where the initial metric perturbation $\Phi = 2\Theta_0$ vanishes, but the particle content (baryons + CDM) is different. Such perturbations exhibit interesting late-time dynamics when the baryons and CDM start to dominate the energy budget of the Universe and hence source metric perturbations. If we modify the baryons, we get a *baryon isocurvature* perturbation:

$$\Theta_0 : \delta_b : \delta_c = 0 : 1 : 0. \tag{40}$$

We could also have a *dark matter isocurvature* perturbation:

$$\Theta_0 : \delta_b : \delta_c = 0 : 0 : 1. \tag{41}$$

A general perturbation is a linear combination of these.

In models with a lepton asymmetry and conserved lepton number, one could have $\mathcal{N}_0 \neq \Theta_0$. This allows for the possibility of a fourth type of perturbation, the *neutrino isocurvature* perturbation:

$$\Theta_0 : \mathcal{N}_0 : \delta_b : \delta_c = -f_\nu : 1 - f_\nu : 0 : 0, \tag{42}$$

in which also initially $\Phi = 0$.

If one has additional fields in the universe (multicomponent dark matter, dynamical dark energy) then of course additional isocurvature perturbations become possible.

Observationally, the adiabatic perturbations explain everything we have seen so far, so we focus on those; but it must be remembered that a small contribution of isocurvature modes is possible.

2 Inflation

At this point, we have a problem: we need a mechanism to generate the perturbations. We have an even more serious problem too, the *horizon problem*: since the patches of CMB sky that we observe today were separated by more than a horizon length at the time of recombination (i.e. their comoving separation is large compared to $\eta_{\rm rec}$), it is not clear why they should be at nearly the same temperature at all.

A nontrivial amount of this section is based on Liddle & Lyth.

Solving the horizon problem. The simplest way to solve this problem is inflation. We propose that the expansion history $H \propto a^{-2}$ did not occur all the way to a singularity at $a = 0$, $T = \infty$, and $H = \infty$. Rather we wish to modify the early expansion history of the Universe to solve the horizon problem. Recall that the conformal time is:

$$\eta = \int \frac{dt}{a} = \int \frac{d\ln a}{aH} = \int \frac{da}{a^2 H}. \tag{43}$$

The horizon problem occurred because this integral is convergent as $a \to 0$. In general suppose that as $a \to 0$ the Universe consisted of some substance with equation of state w. Then we found earlier that:

$$H \propto a^{-3(1+w)/2}. \tag{44}$$

The integration for η then becomes:

$$\eta \propto \int a^{(3w-1)/2}\,da. \tag{45}$$

The integral converges as $a \to 0$ for $w > -1/3$ (e.g. for matter or radiation) and diverges for $w \leq -1/3$. Recall from the Friedmann equation that:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\rho(1+3w). \tag{46}$$

Therefore the "stuff" we need with $w \leq -1/3$ must have sufficiently negative pressure to make the early Universe accelerate – hence the name "inflation." An easy choice would be the cosmological constant Λ with $w = -1$, but we know that Λ is too small. Nevertheless, we can make significant progress by assuming that the Universe went through a phase in which $w \approx -1$. Later we will discuss possible microphysics that leads to this situation.

If indeed inflation took place with $w \approx -1$, there was an energy density ρ_{inf} during inflation and a Hubble constant,

$$H_{inf} = \sqrt{\frac{8}{3}\pi G\rho_{inf}}. \tag{47}$$

The conformal time during inflation was

$$\eta = \int \frac{da}{a^2 H_{inf}} = -\frac{1}{aH_{inf}} + \text{constant}. \tag{48}$$

We need to find the constant of integration now. The simplest possible assumption is that the end of inflation corresponds to the beginning of the hot, radiation-dominated expansion phase. (To go from inflation to the hot phase requires a process known as *reheating* during which the energy density from inflation thermalizes; the conformal time that elapsed during this phase is usually very small.) This corresponds to $\eta \approx 0$ for our usual choice of conformal time.

If inflation ends at scale factor a_e, then we can solve for the integration constant:

$$\eta = \int \frac{da}{a^2 H_{inf}} = -\frac{1}{aH_{inf}} + \frac{1}{a_e H_{inf}}. \tag{49}$$

It is convenient to define a new variable N, the number of e-folds of expansion remaining until the end of inflation:

$$N = \ln\frac{a_e}{a} \quad \leftrightarrow \quad a = a_e e^{-N}. \tag{50}$$

In terms of this variable, we have:

$$\eta = -\frac{1}{a_e H_{inf}}(e^N - 1). \tag{51}$$

The initial conditions we defined earlier for perturbation theory were defined at the surface $\eta = 0$. Two portions of this surface separated by distance r were last in causal contact at $\eta = -r$. Written in terms of the number of e-folds:

$$N = \ln(-a_e H_{inf}\eta - 1) \approx \ln(a_e H_{inf} r). \tag{52}$$

We discuss perturbation modes in Fourier space, so alternatively we can write:

$$N \approx \ln \frac{a_e H_{inf}}{k}. \tag{53}$$

Unfortunately we don't know enough physics to compute N exactly because we don't know a_e or H_{inf}. On the other hand, these two quantities appear in a logarithm so some rough estimation is possible. Suppose first that the Universe reheats instantaneously (which is probably wrong). Then the density at reheating is, at order of magnitude level (leaving out factors of $g_\star$),

$$\rho \sim T^4 \sim T_0^4 a_e^{-4}, \tag{54}$$

where $T_0 = 2.7$ K is the temperature today. By conservation of energy this equals the energy density during inflation ρ_{inf}, so $a_e \sim T_0 \rho_{inf}^{-1/4}$. The Hubble constant during inflation is

$$H_{inf} \sim (G\rho_{inf})^{1/2}. \tag{55}$$

Then:

$$a_e H_{inf} \sim G^{1/2} T_0 \rho_{inf}^{1/4}. \tag{56}$$

Substituting into Eq. (53) gives:

$$N \approx \ln \frac{G^{1/2}\rho_{inf}^{1/4} T_0}{k}. \tag{57}$$

To plug in some typical numbers:

$$N \approx \ln \frac{G^{1/2}\rho_{inf}^{1/4} T_0}{k} = 56 + \ln \frac{\rho_{inf}^{1/4}}{10^{16}\,\mathrm{GeV}} - \ln \frac{k}{1\,\mathrm{Mpc}^{-1}}. \tag{58}$$

This argument could be modified if the Universe went through a non-radiation dominated phase ($w \neq 1/3$) after the end of inflation. An example would be if inflation left behind a nonrelativistic matter field (e.g. coherently oscillating inflaton field) that lived for several e-folds of inflation before decaying into radiation and thermalizing.

The largest scales we observe today in the CMB are of order $k =$few$\times 10^{-4}$ Mpc, which correspond to

$$N \sim 64 + \ln \frac{\rho_{inf}^{1/4}}{10^{16}\,\mathrm{GeV}}. \tag{59}$$

In order for inflation to solve the horizon problem this means it had to have lasted at least ~ 64 e-folds – or maybe somewhat less if the energy density during inflation was low.

Scalar fields. Most models of inflation make use of a scalar field. The action for a minimally coupled scalar field is:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - V(\phi) \right], \tag{60}$$

where $V(\phi)$ is a function called the potential. We can derive the equations of motion for the scalar by taking the functional derivative:

$$0 = \frac{\delta S}{\delta \phi} = \sqrt{-g} \left[\nabla^\mu \nabla_\mu \phi - V'(\phi) \right], \tag{61}$$

so the equation of motion is:

$$\nabla^\mu \nabla_\mu \phi - V'(\phi) = 0. \tag{62}$$

In the case of a homogeneous scalar field in an FRW metric, this simplifies to:

$$\ddot{\phi} + 2aH\dot{\phi} + a^2 V'(\phi) = 0; \tag{63}$$

or, in terms of physical time t,

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V'(\phi) = 0. \tag{64}$$

The equation of state can be obtained from the stress-energy tensor of the scalar field. This is:

$$T_{\mu\nu} = -2\frac{\delta S}{\delta g^{\mu\nu}} = \phi_{;\mu}\phi_{;\nu} - g_{\mu\nu} \left[\frac{1}{2} \phi_{;\alpha} \phi^{;\alpha} + V(\phi) \right]. \tag{65}$$

In the case of the homogeneous scalar field, this gives:

$$\begin{aligned} \rho &= \frac{1}{2} a^{-2} \dot{\phi}^2 + V(\phi); \\ p &= \frac{1}{2} a^{-2} \dot{\phi}^2 - V(\phi). \end{aligned} \tag{66}$$

The equation of state is:

$$w = \frac{\frac{1}{2} a^{-2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} a^{-2} \dot{\phi}^2 + V(\phi)}. \tag{67}$$

This is the basic reason that scalar fields are useful for inflation: if the potential V dominates over the "kinetic" energy term $\dot{\phi}^2/2a^2$, then $w \rightarrow -1$. Then the scalar field can act like a cosmological constant and make the Universe acclerate. But eventually the scalar field can "roll down" to another value of ϕ where $V \approx 0$, and inflation can stop.

Slow-roll conditions. The evolution equation for ϕ (Eq. 64) looks like the equation for a particle moving in a potential with a drag term. Under certain conditions the drag term can dominate over the "inertia" term $d^2\phi/dt^2$, a phenomenon known as slow roll. To see when this may happen, we define the slow-roll parameters:

$$\epsilon(\phi) = \frac{V'^2}{16\pi G V^2}; \quad \bar{\eta}(\phi) = \frac{V''}{8\pi G V}. \tag{68}$$

If slow roll holds and the kinetic energy density is negligible, then

$$\frac{d\phi}{dt} = -\frac{V'(\phi)}{3H} = -\frac{1}{\sqrt{24\pi G}} fracV'(\phi)\sqrt{V(\phi)}. \tag{69}$$

The implied acceleration is obtained by dotting this:

$$\frac{d^2\phi/dt^2}{d\phi/dt} = \frac{d}{dt}\left(\ln\frac{d\phi}{dt}\right) = \frac{V''(\phi)d\phi/dt}{V'(\phi)} - \frac{V'(\phi)d\phi/dt}{2V(\phi)}. \tag{70}$$

Simplify the second term:

$$\frac{d^2\phi/dt^2}{d\phi/dt} = -\frac{aV''(\phi)}{3H} + \frac{[V'(\phi)]^2}{6HV(\phi)}. \tag{71}$$

The ratio of the accleration to the drag is:

$$\frac{d^2\phi/dt^2}{3H\ d\phi/dt} = -\frac{aV''(\phi)}{9H^2} + \frac{[V'(\phi)]^2}{18H^2V(\phi)}. \tag{72}$$

Using the Friedmann equation, $H^2 = (8/3)\pi G V$,

$$\frac{d^2\phi/dt^2}{3H\ d\phi/dt} = -\frac{aV''}{24\pi G V} + \frac{V'^2}{48\pi G V^2}. \tag{73}$$

Thus the second-derivative term can be neglected if ϵ and $|\bar{\eta}|$ are both $\ll 1$. We must also require that kinetic energy density is negligible:

$$\frac{\text{kinetic}}{\text{potential}} = \frac{(d\phi/dt)^2}{2V} = \frac{V'^2}{48\pi G V^2} = \frac{1}{3}\epsilon. \tag{74}$$

Thus if $\epsilon, |\bar{\eta}| \ll 1$ there is a self-consistent slow-roll solution to the equations.

The equation of state during the inflationary epoch is

$$w = \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} = \frac{\epsilon/3 - 1}{\epsilon/3 + 1} = -1 + \frac{2}{3}\epsilon. \tag{75}$$

Thus the slow-roll solution does indeed give us almost-exponential expansion.

The number of e-folds of inflation can be obtained by integrating over the field value (I will take ϕ to move from + to −):

$$N = -\int \frac{d\phi}{d\phi/dN} = -\int H\frac{d\phi}{d\phi/dt} = \int H d\phi V'/3H = \int \frac{3H^2}{V'}d\phi = \int \frac{8\pi G V}{V'}d\phi. \tag{76}$$

The integration constant should be set to $N = 0$ at the end of inflation. This is typically when ϵ becomes large; according to slow-roll, $w = -1/3$ occurs when $\epsilon = 1$, but one should not take this too seriously as slow-roll breaks down at this point.

3 Examples

We now consider some toy models of inflation. These are just the simplest possibilities one can write down; it's possible that inflation was much more complicated (or even that there was an alternative!). But they illustrate the major features that are possible with a single field.

Single massive scalar. The simplest potential one can write down is:

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{77}$$

The slow-roll parameters are:

$$\epsilon(\phi) = \frac{1}{4\pi G\phi^2}; \quad \bar{\eta}(\phi) = \frac{1}{4\pi G\phi^2}. \tag{78}$$

The number of e-folds remaining until the end of inflation is:

$$N = \int \frac{4\pi G m^2\phi^2}{m^2\phi} d\phi = 2\pi G\phi^2, \tag{79}$$

assuming inflation ends when $\phi \to 0$. (Technically it ends when $\phi \sim 1/\sqrt{4\pi G}$ but for $N \gg 1$ this doesn't matter.) Since N relates more directly to observables than ϕ, it is worth writing the slow-roll parameters in terms of N:

$$\epsilon = \bar{\eta} = \frac{1}{2N}. \tag{80}$$

Quartic potential. Another theory, beloved by all QFT students, is:

$$V(\phi) = \frac{1}{24}\lambda\phi^4. \tag{81}$$

Now the slow-roll parameters are

$$\epsilon(\phi) = \frac{1}{\pi G\phi^2}; \quad \bar{\eta}(\phi) = \frac{3}{2\pi G\phi^2}. \tag{82}$$

The number of e-folds remaining until the end of inflation is: (recall $V/V' = 4/\phi$)

$$N = \int 2\pi G\phi d\phi = \pi G\phi^2, \tag{83}$$

and thus:

$$\epsilon = \frac{1}{N} \quad \bar{\eta} = \frac{3}{2N}. \tag{84}$$

Inverted quadratic. Here we try a potential motivated by spontaneous symmetry breaking problems:

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2. \tag{85}$$

The field starts near $\phi = 0$ and rolls toward large ϕ. This potential requires additional terms to come in later (e.g. a $\lambda\phi^4$ term) to stop inflation.

The slow-roll parameters are:

$$\epsilon(\phi) = \frac{\mu^4\phi^2}{16\pi G V_0^2}; \quad \bar{\eta}(\phi) = -\frac{\mu^2}{8\pi G V_0}. \tag{86}$$

(Yes, $\bar{\eta}$ can be negative!) Slow-roll breaks when the scalar field gets too large and the higher-order terms stop it; this is presumably at:

$$\phi_{\rm end} \sim \frac{V_0^{1/2}}{\mu}, \tag{87}$$

although it could happen before then. The number of e-folds is then:

$$N = -\int \frac{8\pi G V_0}{\mu^2\phi}\,d\phi = \frac{8\pi G V_0}{\mu^2}\ln\frac{\phi_{\rm end}}{\phi}. \tag{88}$$

Inverting to get ϕ:

$$\phi = \phi_{\rm end}\exp\frac{-\mu^2 N}{8\pi G V_0} \tag{89}$$

and hence

$$\epsilon = \frac{\mu^4\phi_{\rm end}^2}{16\pi G V_0^2}\exp\frac{-\mu^2 N}{4\pi G V_0} \sim \frac{\mu^2}{16\pi G V_0}\exp\frac{-\mu^2 N}{4\pi G V_0}. \tag{90}$$

This depends on the unknown parameter $\mu^2/16\pi G V_0$, but if $N \sim 60$ then we will need ϵ to be $\ll 1$, and possible $\lll 1$.

Hybrid inflation. This is actually a two-field model of inflation, e.g.:

$$V = V_0 + \frac{1}{2}m^2\phi^2 - \frac{1}{2}m_\psi^2\psi^2 + \frac{1}{4}\lambda\psi^4 + \frac{1}{2}\lambda'\psi^2\phi^2. \tag{91}$$

For large ϕ, the field ψ sits at its minimum $\psi = 0$. But ϕ slides down its potential, and at some critical value

$$\phi_c = \sqrt{\frac{\lambda}{\lambda'}}M, \tag{92}$$

there is a phase transition, the $\psi \to -\psi$ symmetry is spontaneously broken. The two fields then roll together until inflation is terminated.

If the V_0 term dominates the potential, then the slow-roll parameters are:

$$\epsilon(\phi) = \frac{m^4\phi^2}{16\pi G V_0^2}; \quad \bar{\eta}(\phi) = \frac{m^2}{8\pi G V_0}. \tag{93}$$

The number of e-folds of inflation is:

$$N = \int_{\phi_c}^{\phi} \frac{8\pi G V_0}{m^2 \phi}\, d\phi = \frac{8\pi G V_0}{m^2} \ln \frac{\phi}{\phi_c}. \tag{94}$$

The ϵ parameter can be written in terms of N:

$$\epsilon = \frac{m^4 \phi_c^2}{16\pi G V_0^2} \exp \frac{m^2 N}{4\pi G V_0}. \tag{95}$$

Note that in this model, ϵ could be very small if we take ϕ_c to be small. Our freedom to do this will be restricted by the requirement to get the right perturbation spectrum, but we have three knobs to play with (ϕ_c, m, V_0) so a large class of hybrid models is viable.