In the past lectures we constructed the evolution equations for the matter fields. However in order to complete the picture we need to work through the Einstein equations and understand how the spacetime itself evolves.

1 The metric, stress-energy tensor, and gauge transformations.

Metric. As usual, the metric tensor is written as:

$$ds^2 = a^2\left[-(1+2A)d\eta^2 - 2B_i\,dx^i d\eta + (1+2D)dx^i dx^i + 2E_{ij}dx^i dx^j\right]. \quad (1)$$

Stress-energy tensor. The stress-energy tensor, written in the normal frame, is decomposed into a density:

$$\rho = T_{\mu\nu}u^\mu u^\nu = \bar{\rho} + \delta\rho; \quad (2)$$

a 3-momentum density:

$$j_{\hat{i}} = -T_{\mu\nu}u^\mu(\hat{\mathrm{e}}_{\hat{i}})^\nu; \quad (3)$$

a pressure:

$$p = \frac{1}{3}T_{\mu\nu}(\hat{\mathrm{e}}_{\hat{i}})^\mu(\hat{\mathrm{e}}_{\hat{i}})^\nu = \bar{p} + \delta p; \quad (4)$$

and an anisotropic stress:

$$\Sigma_{\hat{i}\hat{j}} = \left(T_{\mu\nu}\hat{\mathrm{e}}^\mu_{\hat{i}}\hat{\mathrm{e}}^\nu_{\hat{j}} - \frac{1}{3}\,p\,\delta_{\hat{i}\hat{j}}\right). \quad (5)$$

Gauge transformations. Often it is possible to simplify a calculation through an appropriate choice of gauge. If one makes a perturbation to the coordinate system,

$$x'^\mu = x^\mu + \xi^\mu, \quad (6)$$

then the metric tensor changes according to:

$$g'_{\mu\nu}(x') = g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}. \quad (7)$$

A general gauge transformation is:

$$\xi^\mu = (T, L^i) \quad \leftrightarrow \quad \xi_\mu = a^2(T, L^i), \quad (8)$$

and its application results in the following changes to the metric:

$$\begin{aligned}
A' &= A - \dot{T} - aHT \\
B'_i &= B_i - T_{,i} - \dot{L}_i \\
D' &= D - aHT - \frac{1}{3}L_{i,i} \\
E'_{ij} &= E_{ij} - \frac{1}{2}(L_{i,j} + L_{j,i}) + \frac{1}{3}L_{k,k}\delta_{ij}.
\end{aligned} \quad (9)$$

If we write this in terms of Fourier modes:

$$
\begin{aligned}
A' &= A - \dot{T} - aHT \\
B_i' &= B_i - \dot{L}_i \quad (i = 1, 2) \\
B_3' &= B_3 - ikT - \dot{L}_3 \\
D' &= D - aHT - \frac{1}{3} ikL_3 \\
E_{11}' - E_{22}' &= E_{11} - E_{22} \\
E_{12}' &= E_{12} \\
E_{i3}' &= E_{i3} - \frac{1}{2} ikL_i \quad (i = 1, 2) \\
E_{33}' &= E_{33} - \frac{2}{3} ikL_3.
\end{aligned}
\tag{10}
$$

Let us assess the 10 pieces of the metric and our ability to remove them with intelligent choices of gauge:

- There are 2 tensorial parts of the metric, $E_{11} - E_{22} \mp 2iE_{12}$. These pick up a factor of $e^{\pm 2i\Delta\phi}$ under rotation by $\Delta\phi$ around the **k**-axis, so they are part of the tensor sector (they couple to $m = \pm 2$ modes in the Boltzmann hierarchy). These are gauge-invariant at first order. These parts will correspond to gravitational waves, and they cannot be gauged away.

- There are 4 scalar parts of the metric ($m = 0$, invariant under rotations around the **k**-axis): A, B_3, D, and E_{33}. None of these is gauge-invariant. There are 2 gauge degrees of freedom here (T and L_3) so we can at most remove 2 of the scalar parts of the metric.

- There are 4 vector ($m = \pm 1$) parts of the metric: $\mp B_1 + iB_2$ and $\mp E_{13} + iE_{23}$ (they pick up a factor of $e^{\pm i\Delta\phi}$ under rotation around the **k**-axis). There are 2 gauge degrees of freedom ($\mp L_1 + iL_2$) so again two of the vector metric components are physical and cannot be removed.

Gauge choices. We need to make a gauge choice for the scalars. Two convenient choices for CMB studies are:

- *Synchronous gauge*: $A = B_3 = 0$. This may be achieved via:

$$
T = a^{-1} \int aA \, d\eta; \quad L_3 = \int (ikT - B_3) \, d\eta. \tag{11}
$$

  The two integration constants are unspecified which means that synchronous gauge does not completely fix the gauge. This is both an asset and a liability. The asset is that for any initial surface (e.g. as predicted by inflation), one can make that surface have constant η by appropriate choice of integration constant. This is a big advantage for numerical codes. On the other hand, the residual gauge freedom complicates analytical investigations. Another property of synchronous gauge is that the trajectories of constant x^i follow geodesics (i.e. are freely falling). Therefore synchronous gauge develops large metric fluctuations even in problems where gravity is nearly Newtonian. For this reason we won't use it in this class.

- *Newtonian gauge*: $B_3 = E_{33} = 0$. This may be achieved via:

$$L_3 = -\frac{3}{2}i\frac{E_{33}}{k}; \quad T = -\frac{i}{k}(B_3 + \dot{L}_3). \tag{12}$$

  There are no constants of integration here so the Newtonian gauge is unique (at least for perturbations of finite wavelength). In class we will use the notation $A = \Psi$ and $D = \Phi$, which is consistent with Dodelson (but in the literature convention varies).

We will use the Newtonian gauge in class since it is used for most analytic work.

For the vectors, the most convenient choice is to take $E_{13} = E_{23} = 0$, which can be achieved by taking:

$$L_i = -2ik^{-1}E_{i3} \quad (i = 1, 2). \tag{13}$$

This way, we have packaged the scalar metric perturbations into $A = \Psi$ and $D = \Phi$, the vectors into B, and the tensors into E.

2 The Einstein equations

The Einstein equations can be obtained by finding the Einstein tensor for our metric, writing it in the normal-frame components, and then taking:

$$G^{\hat{\mu}\hat{\nu}} = 8\pi G T^{\hat{\mu}\hat{\nu}}. \tag{14}$$

This equation has ten components, but in fact we only need six of them because the Bianchi identities combined with conservation of the source (${T^{\mu\nu}}_{;\mu} = 0$) makes four of them redundant. I won't go through the calculation because it is quite mechanical, but the components that we will use are as follows.

Scalar perturbations. For the scalars, the Einstein equations are:

$$\begin{aligned} k^2\Phi + 3aH(\dot{\Phi} - aH\Psi) &= 4\pi G a^2 \delta\rho; \\ \dot{\Phi} - aH\Psi &= 4\pi G a^2 \frac{j_{\hat{3}}}{ik}; \\ aH\dot{\Psi} + 2a(aH^2 + \dot{H})\Psi - \frac{1}{3}k^2\Psi - \ddot{\Phi} - aH\dot{\Phi} &= 4\pi G a^2 \left(\delta p + \frac{1}{3}\delta\rho\right); \\ \frac{2k^2}{3a^2}(\Phi + \Psi) &= 8\pi G \Sigma_{\hat{3}\hat{3}}. \end{aligned} \tag{15}$$

Only two linear combinations of these are independent. (In fact the first two are constraint equations and the other two can be derived by taking their derivatives and using the continuity equation.) The most useful such combinations are the algebraic relations:

$$
\begin{aligned}
\Phi &= \frac{4\pi G a^2}{k^2}\left(\delta\rho + \frac{3iaH}{k} j\right). \\
\Phi + \Psi &= \frac{12\pi G a^2}{k^2}\Sigma_{\hat{3}\hat{3}}.
\end{aligned}
\tag{16}
$$

Vector perturbations. The Einstein equations for the vector perturbations are (for $i = 1, 2$):

$$
\begin{aligned}
k^2 B_i &= 16\pi G a^2 j_{\hat{i}}; \\
\dot{B}_i + 2aHB_i &= -16\pi G a^2 \frac{i}{k}\Sigma_{\hat{i}\hat{3}}.
\end{aligned}
\tag{17}
$$

The first equation is more useful since it is algebraic. (It is the transverse momentum constraint.) The second follows by continuity.

Tensor perturbations. The Einstein equations for the tensors are:

$$
\ddot{E}^{(\pm 2)} + 2aH\dot{E}^{(\pm 2)} + k^2 E^{(\pm 2)} = 8\pi G a^2 \Sigma^{(\pm 2)},
\tag{18}
$$

where we've defined:

$$
\begin{aligned}
E^{(\pm 2)} &= -\frac{1}{\sqrt{6}}(E_{11} - E_{22} \mp 2iE_{12}); \\
\Sigma^{(\pm 2)} &= -\frac{1}{\sqrt{6}}(\Sigma_{\hat{1}\hat{1}} - \Sigma_{\hat{2}\hat{2}} \mp 2i\Sigma_{\hat{1}\hat{2}}).
\end{aligned}
\tag{19}
$$

The key point is that this is a second-order differential equation rather than an algebraic equation. The tensor perturbations $E^{(\pm 2)}$ have dynamics! In fact one can see that they obey a wave equation with a source term given by the anisotropic stress. These perturbations correspond to gravitational waves. The components $E^{(\pm 2)}$ are the right- and left-handed circular polarization amplitudes for the gravitational waves respectively. The circular polarization basis is more convenient for writing the photon hierarchy, but sometimes one writes the linear polarization amplitudes instead:

$$
E_{ij} = \frac{1}{2}\begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix},
\tag{20}
$$

with

$$
E^{(\pm 2)} = -\sqrt{\frac{2}{3}}(h_+ \mp ih_\times).
\tag{21}
$$

Most potential sources of gravitational waves from the early Universe (e.g. inflation) give unpolarized gravitational waves, so either the linear or circular basis is appropriate.

3 Evaluation of stress-energy tensor

In order to complete the cosmological evolution equations, we need to evaluate the stress-energy tensor in terms of the perturbation variables. As long as the constituents of the Universe interact weakly, we may write the total stress-energy tensor as a sum, e.g. for the densities:

$$\delta\rho = \delta\rho_b + \delta\rho_c + \delta\rho_\gamma + \delta\rho_\nu. \tag{22}$$

We have already evaluated the stress-energy tensors for the baryons and dark matter. They have no pressure or anisotropic stress, their density perturbations are simply $\delta\rho_b = \bar\rho_b\delta_b$, and their momentum densities are $j_b = \bar\rho_b v_b$. So we only have to do this exercise for the photons and neutrinos.

For the photons, the density is given by:

$$\begin{aligned}
\rho_\gamma &= \int 2\frac{p^2\,dp\,d^2\hat p^i}{(2\pi)^3} p f(p,\hat p^i) \\
&= \frac{1}{4\pi^3}\int d^2\hat p^i \int \frac{p^3\,dp}{e^{p/[T_\gamma(1+\Theta)]}-1} \\
&= \frac{1}{4\pi^3}\int d^2\hat p^i \frac{\pi^4}{15}[T_\gamma(1+\Theta)]^4 \\
&= \frac{\bar\rho_\gamma}{4\pi}\int d^2\hat p^i (1+4\Theta) \\
&= \bar\rho_\gamma(1+4\Theta_{00}).
\end{aligned} \tag{23}$$

One can try the same strategy for the photon momentum $j_{\hat i}$, which differs by insertion of a factor of $\hat p^i$. We solved this in the last lecture, with the result:

$$j^{\hat i} = -4i\bar\rho_\gamma\left(\frac{-\Theta_{11}+\Theta_{1,-1}}{\sqrt 2}, -i\frac{\Theta_{11}+\Theta_{1,-1}}{\sqrt 2}, \Theta_{10}\right). \tag{24}$$

This can be decomposed into the usual spin ±1 components:

$$j^{(\pm1)} = -4i\bar\rho_\gamma\Theta_{1,\pm1}; \quad j_3 = -4i\bar\rho_\gamma\Theta_{10}. \tag{25}$$

Finally we need the anisotropic stress. The scalar component is:

$$\begin{aligned}
\Sigma_\gamma^{\hat 3\hat 3} &= \int 2\frac{p^2\,dp\,d^2\hat p^i}{(2\pi)^3} p\left[(\hat p^3)^2 - \frac13\right] f(p,\hat p^i) \\
&= \frac{1}{4\pi^3}\int d^2\hat p^i \left[(\hat p^3)^2 - \frac13\right]\int \frac{p^3\,dp}{e^{p/[T_\gamma(1+\Theta)]}-1} \\
&= \frac{\bar\rho_\gamma}{4\pi}\int d^2\hat p^i \left[(\hat p^3)^2 - \frac13\right](1+4\Theta).
\end{aligned} \tag{26}$$

Now the quantity in brackets is

$$\sqrt{\frac{16\pi}{45}} Y^*_{20}(\hat{\mathbf p}). \tag{27}$$

Therefore we can collapse the last integral to the 20 multipole of the photon moments:

$$\Sigma_\gamma^{\hat{3}\hat{3}} = -\frac{8}{3}\bar{\rho}_\gamma \Theta_{20}. \tag{28}$$

We also need the tensor components $\Sigma_\gamma^{(\pm 2)}$. These are the same as $\Sigma_\gamma^{\hat{3}\hat{3}}$, but with the replacement:

$$(\hat{p}^3)^2 - \frac{1}{3} \to -\frac{1}{\sqrt{6}}[(\hat{p}^1)^2 - (\hat{p}^2)^2 \mp 2i\hat{p}^1\hat{p}^2]. \tag{29}$$

In terms of spherical harmonics, we make the replacement:

$$\sqrt{\frac{16\pi}{45}} Y_{20}^*(\hat{\mathbf{p}}) \to -\sqrt{\frac{16\pi}{45}} Y_{2,\pm 2}^*(\hat{\mathbf{p}}). \tag{30}$$

This leads us to the result

$$\Sigma_\gamma^{(\pm 2)} = \frac{8}{3}\bar{\rho}_\gamma \Theta_{2,\pm 2}. \tag{31}$$

Similar equations apply to the neutrinos. The stress-energy terms that we need are thus:

$$\begin{aligned}
\delta\rho &= \bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c + 4\bar{\rho}_\gamma\Theta_{00} + 4\bar{\rho}_\nu\mathcal{N}_{00};\\
j_3 &= \bar{\rho}_b v_b^3 + \bar{\rho}_c v_c^3 - 4i\bar{\rho}_\gamma\Theta_{10} - 4i\bar{\rho}_\nu\mathcal{N}_{10};\\
j^{(\pm 1)} &= \bar{\rho}_b v_b^{(\pm 1)} + \bar{\rho}_c v_c^{(\pm 1)} - 4i\bar{\rho}_\gamma\Theta_{1,\pm 1} - 4i\bar{\rho}_\nu\mathcal{N}_{1,\pm 1};\\
\Sigma^{\hat{3}\hat{3}} &= -\frac{8}{3}\bar{\rho}_\gamma\Theta_{20} - \frac{8}{3}\bar{\rho}_\nu\mathcal{N}_{20};\\
\Sigma^{(\pm 2)} &= -\frac{8}{3}\bar{\rho}_\gamma\Theta_{2,\pm 2} - \frac{8}{3}\bar{\rho}_\nu\mathcal{N}_{2,\pm 2}.
\end{aligned} \tag{32}$$

4 Overall equations

We are now in a position to write down the full set of perturbation equations (finally!)

Scalars.

$$\begin{aligned}
\dot{\Theta}_0 &= -k\Theta_1 - \dot{\Phi};\\
\dot{\Theta}_1 &= k\frac{\Theta_0 - 2\Theta_2}{3} + \dot{\tau}\Theta_1 - \frac{1}{3}i\dot{\tau}v_{b3} + \frac{1}{3}k\Psi;\\
\dot{\Theta}_2 &= k\frac{2\Theta_1 - 3\Theta_3}{5} + \frac{9}{10}\dot{\tau}\Theta_2;\\
\dot{\Theta}_l &= k\left[\frac{l}{2l+1}\Theta_{l-1} - \frac{l+1}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \ \ (l \geq 3);\\
\dot{\mathcal{N}}_0 &= -k\mathcal{N}_1 - \dot{\Phi};\\
\dot{\mathcal{N}}_1 &= k\frac{\mathcal{N}_0 - 2\mathcal{N}_2}{3} + \frac{1}{3}k\Psi;
\end{aligned}$$

$$
\begin{aligned}
\dot{\mathcal{N}}_l &= k\left[\frac{l}{2l+1}\mathcal{N}_{l-1} - \frac{l+1}{2l+1}\mathcal{N}_{l+1}\right] \quad (l \geq 2);\\
\dot{\delta}_c &= -3\dot{\Phi} - ikv_c;\\
\dot{v}_c &= -aHv_c - ik\Psi\\
\dot{\delta}_b &= -3\dot{\Phi} - ikv_b;\\
\dot{v}_b &= -aHv_b - ik\Psi + \frac{\dot{\tau}}{R}(v_b + 3i\Theta_1)\\
k^2\Phi + 3aH(\dot{\Phi} - aH\Psi) &= 4\pi Ga^2(\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c + 4\bar{\rho}_\gamma\Theta_0 + 4\bar{\rho}_\nu\mathcal{N}_0);\\
\Phi &= \frac{4\pi Ga^2}{k^2}\left[\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c + 4\bar{\rho}_\gamma\Theta_0 + 4\bar{\rho}_\nu\mathcal{N}_0 + \frac{3iaH}{k}\left(\bar{\rho}_b v_b + \bar{\rho}_c v_c - 4i\bar{\rho}_\gamma\Theta_1 - 4i\bar{\rho}_\nu\mathcal{N}_1\right)\right];\\
\Phi + \Psi &= -\frac{32\pi Ga^2}{3k^2}\left(\rho_\gamma\Theta_2 + \bar{\rho}_\nu\mathcal{N}_2\right). \qquad (33)
\end{aligned}
$$

(Of the last three equations, one is redundant but we will use all three of them at various times.)

Vectors. Suppressing the $m = \pm 1$ indices:

$$
\begin{aligned}
\dot{\Theta}_1 &= -\frac{1}{\sqrt{3}}k\Theta_2 + \dot{\tau}\Theta_1 - \frac{i\dot{\tau}}{3}v_b\\
\dot{\Theta}_2 &= k\frac{\sqrt{3}\Theta_1 - 2\sqrt{2}\Theta_3}{5} + \frac{iB}{5\sqrt{3}} + \frac{9}{10}\dot{\tau}\Theta_2;\\
\dot{\Theta}_l &= k\left[\frac{\sqrt{(l-1)(l+1)}}{2l+1}\Theta_{l-1} - \frac{\sqrt{l(l+2)}}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \quad (l \geq 3);\\
\dot{\mathcal{N}}_1 &= -\frac{1}{\sqrt{3}}k\mathcal{N}_2;\\
\dot{\mathcal{N}}_2 &= k\frac{\sqrt{3}\mathcal{N}_1 - 2\sqrt{2}\mathcal{N}_3}{5} + \frac{iB}{5\sqrt{3}};\\
\dot{\mathcal{N}}_l &= k\left[\frac{\sqrt{(l-1)(l+1)}}{2l+1}\mathcal{N}_{l-1} - \frac{\sqrt{l(l+2)}}{2l+1}\mathcal{N}_{l+1}\right] \quad (l \geq 3);\\
\dot{v}_c &= -aHv_c;\\
\dot{v}_b &= -aHv_b + \frac{\dot{\tau}}{R}[v_b + 3i\Theta_1];\\
k^2 B &= 16\pi Ga^2(\bar{\rho}_b v_b + \bar{\rho}_c v_c - 4i\bar{\rho}_\gamma\Theta_1 - 4i\bar{\rho}_\nu\mathcal{N}_1). \qquad (34)
\end{aligned}
$$

(Note that there are no density perturbations since the vector perturbations are transverse.)

Tensors. Again suppressing the $m = \pm 2$ indices:

$$
\begin{aligned}
\dot{\Theta}_2 &= -\frac{1}{\sqrt{5}}k\Theta_3 + \frac{9}{10}\dot{\tau}\Theta_2 - \frac{\dot{E}}{5};\\
\dot{\Theta}_l &= k\left[\frac{\sqrt{(l-2)(l+2)}}{2l+1}\Theta_{l-1} - \frac{\sqrt{(l-1)(l+3)}}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \quad (l \geq 3)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathcal{N}}_2 &= -\frac{1}{\sqrt{5}} k \mathcal{N}_3 - \dot{E}; \\
\dot{\mathcal{N}}_l &= k \left[\frac{\sqrt{(l-2)(l+2)}}{2l+1} \mathcal{N}_{l-1} - \frac{\sqrt{(l-1)(l+3)}}{2l+1} \mathcal{N}_{l+1} \right] \quad (l \geq 3); \\
\ddot{E} + 2aH\dot{E} + k^2 E &= -\frac{32}{3} \pi G a^2 (\bar{\rho}_\gamma \Theta_2 + \bar{\rho}_\nu \mathcal{N}_2).
\end{aligned}
\tag{35}
$$