1 Objectives

In this lecture we will take the photon multipole equations derived last time, and convert them into Fourier-multipole space. This will be convenient for linear perturbation theory, since each Fourier mode evolves independently, and for the CMB, since the superposition of CMB fluctuations is simpler in multipole than in angle space.

2 Fourier transform

We wrote down the Boltzmann equation for photons last time; it is:

$$\begin{aligned}\dot{\Theta} &= -\hat{p}^i\frac{\partial\Theta}{\partial x^i} - \hat{p}^i\frac{\partial A}{\partial x^i} - \hat{p}^i\hat{p}^j\frac{\partial B_i}{\partial x^j} - \dot{D} - \hat{p}^i\hat{p}^j\dot{E}_{ij} \\ &\quad -an_e\sigma_T(\Theta - \mathbf{v}_b\cdot\hat{\mathbf{p}}) + \frac{3}{16\pi}an_e\sigma_T\int[1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2]\Theta(\hat{p}'^i)d^2\hat{p}'^i. \qquad (1)\end{aligned}$$

We will now take the Fourier transform of all the perturbation variables. For example,

$$\Theta(\mathbf{k},\hat{\mathbf{p}},\eta) = \int_{\mathbf{R}^3}\Theta(x^i,\hat{\mathbf{p}},\eta)e^{-ik_ix^i}\,d^3x^i. \qquad (2)$$

The advantage of this approach is that the partial derivative operator $\partial/\partial x^i$ simply becomes multiplication: ik_i. We can therefore transform Eq. (1) into:

$$\begin{aligned}\dot{\Theta} &= -ik_i\hat{p}^i\Theta - ik_i\hat{p}^iA - ik_j\hat{p}^i\hat{p}^jB_i - \dot{D} - \hat{p}^i\hat{p}^j\dot{E}_{ij} \\ &\quad +\dot{\tau}(\Theta - \mathbf{v}_b\cdot\hat{\mathbf{p}}) - \frac{3}{16\pi}\dot{\tau}\int[1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2]\Theta(\hat{p}'^i)d^2\hat{p}'^i. \qquad (3)\end{aligned}$$

(Definition: $\dot{\tau} = -an_e\sigma_T$ is the optical depth per unit conformal time – this is negative so that we will later be able to define τ as the optical depth from the observer to a particular redshift.) In this equation, Θ, A, B_i, D, E_{ij}, and $\mathbf{v}_b$ have been Fourier-transformed. General rule: perturbation variables are Fourier-transformed; backgrounds (e.g. n_e) or independent variables (e.g. $\hat{p}^i$) are not. Note that there are no terms that couple different Fourier modes, as appropriate for a homogeneous background.

3 Multipole decomposition

A further simplification is possible if we decompose the perturbations in multipoles. We begin by switching to a coordinate system in whick $\mathbf{k}$ points along the 3-axis. (Superposing different Fourier modes will be necessary to get e.g. the CMB power spectrum, and will involve some bookkeeping and rotation matrices, but no new physics.) So we can simply write $k\hat{\mathbf{e}}_3$ in place of $\mathbf{k}$.

In this coordinate system, one can write $\hat{\mathbf{p}}$ in spherical coordinates:

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad (4)$$

with $\mu = \cos\theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}$. We will then do a multipole decomposition:

$$\Theta(\mathbf{k}, \theta, \phi, \eta) = \sum_{lm} (-i)^l \sqrt{4\pi(2l+1)}\Theta_{lm}(\mathbf{k}, \eta) Y_{lm}(\theta, \phi), \tag{5}$$

which has inverse:

$$\Theta_{lm}(\mathbf{k}, \eta) = \frac{i^l}{\sqrt{4\pi(2l+1)}} \int Y_{lm}^*(\theta, \phi)\Theta(\mathbf{k}, \theta, \phi, \eta) \sin\theta \, d\theta \, d\phi. \tag{6}$$

In the special case of $m = 0$ we may write this in terms of the Legendre polynomials:

$$\Theta_{l0}(\mathbf{k}, \eta) = \frac{i^l}{4\pi} \int P_l(\mu)\Theta(\mathbf{k}, \theta, \phi, \eta) d\mu \, d\phi. \tag{7}$$

The advantage of the multipole decomposition is that it dramatically simplifies Eq. (3). Taking the time derivative of Eq. (6), we get:

$$\dot{\Theta}_{lm}(\mathbf{k}, \eta) = \frac{i^l}{\sqrt{4\pi(2l+1)}} \int Y_{lm}^*(\theta, \phi)\dot{\Theta}(\mathbf{k}, \theta, \phi, \eta) \sin\theta \, d\theta \, d\phi. \tag{8}$$

Therefore if we evaluate each term in the Boltzmann equation, and substitute into Eq. (8), we can get the multipole-space Boltzmann equation. This is our next goal.

Free-streaming term. First let's consider the free-streaming term, $-ik_i\hat{p}^i\Theta$. We can simplify $k_i\hat{p}^i = k\mu$. So the contribution to $\dot{\Theta}_{lm}$ from this term is, from Eq. (8):

$$\begin{aligned}
\dot{\Theta}_{lm}|_{\rm fs} &= \frac{i^l}{\sqrt{4\pi(2l+1)}} \int Y_{lm}^*(\theta, \phi)(-ik\mu)\Theta(\theta, \phi) \sin\theta \, d\theta \, d\phi \\
&= -ik\mu \frac{i^l}{\sqrt{4\pi(2l+1)}} \int Y_{lm}^*(\theta, \phi) \sum_{l'm'} (-i)^{l'} \sqrt{4\pi(2l'+1)}\Theta_{l'm'} Y_{l'm'}(\theta, \phi) \sin\theta \, d\theta \, d\phi \\
&= -ik\mu \sum_{l'm'} i^{l-l'} \sqrt{\frac{2l'+1}{2l+1}} \Theta_{l'm'} \int Y_{lm}^*(\theta, \phi)\mu Y_{l'm'}(\theta, \phi) \sin\theta \, d\theta \, d\phi. \qquad (9)
\end{aligned}$$

We've simplified the problem down to an integral over spherical harmonics. This integral comes with a simple selection rule, namely that by azimuthal symmetry in ϕ it is only nonzero if $m' = m$. It is also straightforward to see that under the symmetry $\theta \to \pi - \theta$ ($mu \to -\mu$) we must have $l - l'$ odd. Finally, μ is a spherical harmonic of order 1, so the triangle rule for addition of angular momenta implies $l - l' = -1, 0, +1$ (and we just learned that 0 is ruled out). So all of this boils down to the two cases of $l' = l - 1$, $m' = m$ and $l' = l + 1$, $m' = m$. The relevant integral is (recall dipole radiation from atomic physics!):

$$\int Y_{lm}^*(\theta, \phi)\mu Y_{l-1,m}(\theta, \phi) \sin\theta \, d\theta \, d\phi = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}. \tag{10}$$

By swapping l and $l-1$ we may obtain the other integral,

$$\int Y^*_{lm}(\theta,\phi)\mu Y_{l+1,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sqrt{\frac{(l+1+m)(l+1-m)}{(2l+3)(2l+1)}}. \tag{11}$$

Putting it all together:

$$\dot{\Theta}_{lm}|_{\rm fs} = k\left[\frac{\sqrt{(l+m)(l-m)}}{2l+1}\Theta_{l-1,m} - \frac{\sqrt{(l+1+m)(l+1-m)}}{2l+1}\Theta_{l+1,m}\right]. \tag{12}$$

Gravitational sources. We next consider the terms in the Boltzmann equation from the metric perturbations A, B, D, and E. Take A as an example:

$$\dot{\Theta}_{lm}|_A = \frac{i^l}{\sqrt{4\pi(2l+1)}}\int Y^*_{lm}(\theta,\phi)(-ik\mu A)\sin\theta\,d\theta\,d\phi. \tag{13}$$

But we can factor out $-ikA$ and decompose μ as a spherical harmonic,

$$\mu = \cos\theta = \sqrt{\frac{4\pi}{3}}\,Y_{10}(\theta,\phi), \tag{14}$$

so that orthonormality implies:

$$\dot{\Theta}_{lm}|_A = \frac{1}{3}kA\delta_{l1}\delta_{m0}. \tag{15}$$

Thus the A metric perturbation only sources a photon dipole, and even then only the $m=0$ component.

The same exercise can be repeated for the other gravitational source terms B, D, and E. The overall result is:

$$\begin{aligned}
\dot{\Theta}_{00}|_{\rm grav} &= -\frac{1}{3}ikB_3 - \dot{D};\\
\dot{\Theta}_{10}|_{\rm grav} &= \frac{1}{3}kA;\\
\dot{\Theta}_{1,\pm1}|_{\rm grav} &= 0;\\
\dot{\Theta}_{20}|_{\rm grav} &= \frac{2}{15}ikB_3 + \frac{1}{5}\dot{E}_{33};\\
\dot{\Theta}_{2,\pm1}|_{\rm grav} &= -\frac{1}{5\sqrt{6}}ik(\pm B_1 - iB_2) - \frac{1}{5\sqrt{6}}(\pm\dot{E}_{13} - i\dot{E}_{23});\\
\dot{\Theta}_{2,\pm2}|_{\rm grav} &= \frac{1}{5\sqrt{6}}(\dot{E}_{11} - \dot{E}_{22} \mp 2i\dot{E}_{12});\\
\dot{\Theta}_{lm,l\geq3}|_{\rm grav} &= 0.
\end{aligned} \tag{16}$$

The fact that there are only local sources for $l \leq 2$ is a direct consequnce of gravity being a spin-2 field.

Scattering terms. The scattering term has three parts. The first (S1) is the $\dot\tau\Theta$ term, which trivially tansforms into multipole space as:

$$\dot\Theta_{lm}|_{\rm S1} = \dot\tau\Theta_{lm}. \tag{17}$$

The second (S2) is the baryon velocity term $-\dot\tau\mathbf{v}_b\cdot\hat{\mathbf{p}}$, which can be transformed the same way we handled the gravitational sources:

$$\begin{aligned}\dot\Theta_{10}|_{\rm S2} &= -\frac{1}{3}i\dot\tau v_{b3};\\ \dot\Theta_{1,\pm1}|_{\rm S2} &= -\frac{1}{3\sqrt{2}}i\dot\tau(\mp v_{b1}+iv_{b2}).\end{aligned} \tag{18}$$

(All others vanish since velocity is a dipole.)

The third term (S3) is:

$$\dot\Theta_{lm}|_{\rm S3} = -\frac{i^l}{\sqrt{4\pi(2l+1)}}\int\int Y^*_{lm}(\hat{\mathbf{p}})\frac{3}{16\pi}\dot\tau[1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2]\Theta(\hat{\mathbf{p}}')d^2\hat{\mathbf{p}}'d^2\hat{\mathbf{p}}. \tag{19}$$

Its evaluation involves expanding the dot product, and separating out the $\hat{\mathbf{p}}$ integral. This will be a homework exercise, but the answer is:

$$\begin{aligned}\dot\Theta_{00}|_{\rm S3} &= -\dot\tau\Theta_{00};\\ \dot\Theta_{2m}|_{\rm S3} &= -\frac{1}{10}\dot\tau\Theta_{2m},\end{aligned} \tag{20}$$

with all others vanishing.

Scalars, vectors, and tensors. Note that the different values of m don't talk to each other; this is a consequence of rotational invariance around the $\mathbf{k}$ axis. So we can consider each case independently. We'll focus on the $m=0$ modes, i.e. perturbations that are invariant under rotations around the $\mathbf{k}$ axis. These are called *scalar* modes in cosmology, which distinguishes them from *vector* modes ($m=\pm1$) and *tensor* modes ($m=\pm2$). The scalars are the only primordial modes that have been detected; we'll discuss tensors later but not vectors (in standard cosmology they have no source).

The scalar modes. If we put all of this together, we get for the $m=0$:

$$\begin{aligned}\dot\Theta_{00} &= -k\Theta_{10}-\frac{1}{3}ikB_3-\dot D;\\ \dot\Theta_{10} &= k\frac{\Theta_{00}-2\Theta_{20}}{3}+\dot\tau\Theta_{10}-\frac{1}{3}i\dot\tau v_{b3}+\frac{1}{3}kA;\\ \dot\Theta_{20} &= k\frac{2\Theta_{10}-3\Theta_{30}}{5}+\frac{9}{10}\dot\tau\Theta_{20}+\frac{2}{15}ikB_3+\frac{1}{5}\dot E_{33};\\ \dot\Theta_{l0} &= k\left[\frac{l}{2l+1}\Theta_{l-1,0}-\frac{l+1}{2l+1}\Theta_{l+1,0}\right]+\dot\tau\Theta_{l0}\;\;(l\ge3).\end{aligned} \tag{21}$$

The usual computation occurs in the Newtonian gauge ($A=\Psi$, $D=\Phi$, $B=E=0$) in which case, dropping the 0 subscripts:

$$\dot\Theta_0 = -k\Theta_1-\dot\Psi;$$

$$
\begin{aligned}
\dot{\Theta}_1 &= k\frac{\Theta_0 - 2\Theta_2}{3} + \dot{\tau}\Theta_1 - \frac{1}{3}i\dot{\tau}v_{b3} + \frac{1}{3}k\Psi; \\
\dot{\Theta}_2 &= k\frac{2\Theta_1 - 3\Theta_3}{5} + \frac{9}{10}\dot{\tau}\Theta_2; \\
\dot{\Theta}_l &= k\left[\frac{l}{2l+1}\Theta_{l-1} - \frac{l+1}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \;\; (l \geq 3).
\end{aligned} \tag{22}
$$

The sequence of multipole moments is called the *Boltzmann hierarchy.*

The vector modes. One can write a similar hierarchy for the vector modes, $m = \pm 1$. In this case there is no monopole ($l = 0$). One finds:

$$
\begin{aligned}
\dot{\Theta}_1 &= -\frac{1}{\sqrt{3}}k\Theta_2 + \dot{\tau}\Theta_1 - \frac{i\dot{\tau}}{3\sqrt{2}}(\mp v_{b1} + iv_{b2}); \\
\dot{\Theta}_2 &= k\frac{\sqrt{3}\Theta_1 - 2\sqrt{2}\Theta_3}{5} - \frac{\pm i(B_1 - i\dot{E}_{13}) + (B_2 - i\dot{E}_{23})}{5\sqrt{6}} + \frac{9}{10}\dot{\tau}\Theta_2; \\
\dot{\Theta}_l &= k\left[\frac{\sqrt{(l-1)(l+1)}}{2l+1}\Theta_{l-1} - \frac{\sqrt{l(l+2)}}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \;\; (l \geq 3).
\end{aligned} \tag{23}
$$

Inflation does not generate vector perturbations, and they are decaying anyway (as we'll see later – it requires us to understand baryon velocity). So we won't do much with them. In exotic scenarios (cosmic strings) one can have vectors.

The tensor modes. Finally we consider the tensor modes, $m = \pm 2$. These have no monopole or dipole. They cannot be generated in linear perturbation theory by density fluctuations, but one can make them during inflation. (Primordial gravitational waves!) The equations are:

$$
\begin{aligned}
\dot{\Theta}_2 &= -\frac{1}{\sqrt{5}}k\Theta_3 + \frac{9}{10}\dot{\tau}\Theta_2 + \frac{1}{5\sqrt{6}}(\dot{E}_{11} - \dot{E}_{22} \mp 2i\dot{E}_{12}); \\
\dot{\Theta}_l &= k\left[\frac{\sqrt{(l-2)(l+2)}}{2l+1}\Theta_{l-1} - \frac{\sqrt{(l-1)(l+3)}}{2l+1}\Theta_{l+1}\right] + \dot{\tau}\Theta_l \;\; (l \geq 3)
\end{aligned} \tag{24}
$$

4 Neutrino equations

Neutrinos are just like photons with two exceptions: (i) they are fermions, and (ii) they don't have Thomson scattering. (Actually they have mass but that doesn't concern us yet.)

For a fermion that was initially in a Fermi-Dirac distribution with zero chemical potential, the unperturbed phase space density is:

$$
f^{(0)}(x^i, p, \hat{p}^i; \eta) = \frac{1}{e^{p/T_\nu} + 1}. \tag{25}
$$

In analogy to the photons, we can define a neutrino temperature perturbation $\mathcal{N}$:

$$
f(x^i, p, \hat{p}^i; \eta) = \left\{\exp\frac{p}{T_\nu(\eta)[1 + \mathcal{N}(x^i, p, \hat{p}^i; \eta)]} + 1\right\}^{-1}. \tag{26}
$$

If we had detectors that could see milli-eV neutrinos then we could make maps of $\mathcal{N}$ just as the CMB observers make maps of Θ.

One can go through the machinery of Fourier-transforming $\mathcal{N}$ and then doing a multipole decomposition. It works the same way as for photons, except that it is simpler (no scattering term). We get, for the scalars in Newtonian gauge:

$$\begin{aligned} \dot{\mathcal{N}}_0 &= -k\mathcal{N}_1 - \dot{\Psi}; \\ \dot{\mathcal{N}}_1 &= k\frac{\mathcal{N}_0 - 2\mathcal{N}_2}{3} + \frac{1}{3}k\Psi; \\ \dot{\mathcal{N}}_l &= k\left[\frac{l}{2l+1}\mathcal{N}_{l-1} - \frac{l+1}{2l+1}\mathcal{N}_{l+1}\right] \quad (l \geq 2), \end{aligned} \tag{27}$$

etc.

5 Dark matter

Next we come to the dark matter. Usually in cosmology we will assume that it is cold dark matter (CDM), which means that initially the dark matter particles all move at the same velocity. Late in the history of the Universe when galaxies form, one may have CDM particles whose orbits cross and hence at a given point there may be a velocity dispersion, but this is not part of linear perturbation theory.

There was at one time a theory of hot dark matter (HDM) where initially the dark matter particles were moving rapidly ("rapidly" = can move a perturbation wavelength in less than the age of the universe). We could do this but it's very complicated.

So here we go with CDM: the dark matter at each point is described by a density ρ_c and a velocity $\mathbf{v}_c$ (subscript $_c$ is for CDM). We can determine their equations of motion from the continuity equation,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{28}$$

So all we need to do is write $T^{\mu\nu}$ in terms of ρ_c and $\mathbf{v}_c$, and we're done.

The stress-energy tensor in the normal observer's frame, is:

$$T^{\hat{0}\hat{0}} = \rho_c; \quad T^{\hat{0}\hat{i}} = \rho_c v_c^i; \quad T^{\hat{i}\hat{j}} = 0. \tag{29}$$

We can convert this into a contravariant tensor by:

$$T^{\mu\nu} = T^{\hat{0}\hat{0}}u^\mu u^\nu + T^{\hat{0}\hat{i}}[u^\mu(\mathbf{e}_{\hat{i}})^\nu + u^\nu(\mathbf{e}_{\hat{i}})^\mu] + T^{\hat{i}\hat{j}}(\mathbf{e}_{\hat{i}})^\mu(\mathbf{e}_{\hat{j}})^\nu. \tag{30}$$

Recall that $u^\mu = a^{-1}(1-A, B_i)$; then:

$$\begin{aligned} T^{00} &= \frac{\rho_c}{a^2}(1-2A); \\ T^{0i} &= \frac{\rho_c}{a^2}(B_i + v_c^i); \\ T^{ij} &= 0. \end{aligned} \tag{31}$$

Going back to the continuity equation, we can write the $\nu = 0$ component:

$$\dot{T}^{00} + \partial_i T^{0i} + \Gamma^{\mu}_{\mu 0} T^{00} + \Gamma^{\mu}_{\mu i} T^{0i} + \Gamma^{0}_{00} T^{00} + 2\Gamma^{0}_{0i} T^{0i} + 2\Gamma^{0}_{ij} T^{ij} = 0. \tag{32}$$

In first order perturbation theory, $T^{ij} = 0$ so we drop the last term. Also T^{0i}, Γ^0_{0i}, and $\Gamma^{\mu}_{\mu i}$ are all first order, so their products are second-order and can be dropped:

$$\dot{T}^{00} + \partial_i T^{0i} + (\Gamma^{\mu}_{\mu 0} + \Gamma^{0}_{00}) T^{00} = 0. \tag{33}$$

We can now use the Christoffel symbols:

$$\Gamma^{\mu}_{\mu 0} = \partial_0 \ln \sqrt{|g|} = 4aH + \dot{A} + 3\dot{D}; \quad \Gamma^{0}_{00} = aH + \dot{A}. \tag{34}$$

Then we have:

$$\frac{\rho_c}{a^2}\left\{-2\dot{A} + [\frac{\dot{\rho}_c}{\rho_c} - 2aH](1-2A) + \partial_i(B_i + v_c^i) + (5aH + 2\dot{A} + 3\dot{D})(1-2A)\right\} = 0. \tag{35}$$

Simplifying:

$$(1-2A)\frac{\dot{\rho}_c}{\rho_c} + \partial_i(B_i + v_c^i) + 3aH(1-2A) + 3\dot{D} = 0. \tag{36}$$

Divide by $1 - 2A$ and solve for $\dot{\rho}_c$:

$$\frac{\dot{\rho}_c}{\rho_c} = -3(aH + \dot{D}) - \partial_i(B_i + v_c^i). \tag{37}$$

Note that in the unperturbed case, this is $-3aH$, in accordance with the usual scaling $\rho_c \propto a^{-3}$.

It is conventional to define the fractional density perturbation,

$$\delta_c = \frac{\rho_c}{\rho_c^{(0)}} - 1. \tag{38}$$

Its derivative is, to first order,

$$\dot{\delta}_c = \partial_\eta \left(\ln \frac{\delta_c}{\delta_c^{(0)}}\right) = \partial_\eta \ln \delta_c - \partial_\eta \ln \delta_c^{(0)}. \tag{39}$$

The second term is $-3aH$, so we get:

$$\dot{\delta}_c = -3\dot{D} - \partial_i(B_i + v_c^i). \tag{40}$$

Now we need to get an equation for the dark matter velocity. This comes from momentum conservation, i.e. the $\nu = i$ components of $\nabla_\mu T^{\mu\nu} = 0$:

$$\dot{T}^{0i} + \partial_j T^{ij} + \Gamma^{\mu}_{\mu 0} T^{0i} + \Gamma^{\mu}_{\mu j} T^{ji} + \Gamma^{i}_{00} T^{00} + 2\Gamma^{i}_{0j} T^{0j} + \Gamma^{i}_{jk} T^{jk} = 0. \tag{41}$$

Again, the purely spacelike components of T vanish at first order, so we have:

$$\dot{T}^{0i} + \Gamma^{\mu}_{\mu 0} T^{0i} + \Gamma^{i}_{00} T^{00} + 2\Gamma^{i}_{0j} T^{0j} = 0. \tag{42}$$

Now T^{0j} is a first-order quantity, so we only need $\Gamma^{\mu}_{\mu 0}$ and Γ^{i}_{0j} to zeroeth order:

$$\Gamma^{\mu}_{\mu 0} = 4aH; \quad \Gamma^{i}_{0j} = aH\delta^{i}_{j}. \tag{43}$$

The other Christoffel symbol we need to first order:

$$\Gamma^{i}_{00} = A_{,i} - \dot{B}_i - aHB_i. \tag{44}$$

With this we find:

$$\frac{\rho_c}{a^2}\left\{\left(-2aH + \frac{\dot{\rho}_c}{\rho_c}\right)(B_i + v^i_c) + \dot{B}_i + \dot{v}^i_c + 4aH(B_i + v^i_c) + A_{,i} - \dot{B}_i - aHB_i + 2aH(B_i + v^i_c)\right\} = 0. \tag{45}$$

Now $\dot{\rho}_c/\rho_c$ multiplies a perturbation so we can substitute the unperturbed value $-3aH$. This simplifies our equation to:

$$aHv^i_c + \dot{v}^i_c + A_{,i} = 0, \tag{46}$$

or:

$$\dot{v}^i_c = -aHv^i_c - A_{,i}. \tag{47}$$

Fourier decomposition. If we take the Fourier transform of the density and velocity equations, we get:

$$\begin{aligned} \dot{\delta}_c &= -3\dot{D} - ik(B_3 + v^3_c); \\ \dot{v}^3_c &= -aHv^3_c - ikA; \\ \dot{v}^i_c &= -aHv^i_c \ \ (i = 1, 2). \end{aligned} \tag{48}$$

In the case of the photons and neutrinos, we decomposed the perturbations into scalars (invariant under rotations around the 3 axis) and vectors (which pick up a factor of $e^{\pm i\Delta\phi}$ under such rotations). In this case δ_c and v^3_c are scalars. The vectors are:

$$v^{(\pm 1)}_c = \frac{\mp v^1_c + iv^2_c}{\sqrt{2}}. \tag{49}$$

The numbers $v^{(+1)}_c$, v^3_c, and $v^{(-1)}_c$ describe the 3 components of velocity. The equations of motion for the vectors are:

$$\dot{v}^{(\pm 1)}_c = -aHv^{(\pm 1)}_c, \tag{50}$$

which have solution $v^{(\pm 1)}_c \propto a^{-1}$. This means that the vector perturbations in the CDM are purely decaying and (aside from nonlinear effects) are not expected today.

In the Newtonian gauge, the scalar equations are:

$$\begin{aligned} \dot{\delta}_c &= -3\dot{\Phi} - ikv_c; \\ \dot{v}_c &= -aHv_c - ik\Psi. \end{aligned} \tag{51}$$

6 Baryons

Cold baryons. Finally we come to the baryonic matter. We will treat the baryons can be treated as pressureless ("cold") just like the dark matter. The condition under which this is justified is that a pressure (sound) wave not be able to propagate a perturbation wavelength in a Hubble time, i.e.

$$\frac{c_s}{aH} \ll k^{-1}. \tag{52}$$

We must evaluate the left hand side in order to establish the range of scales over which the baryons may be treated as cold. At recombination ($z = 1100$), the temperature of the baryons is 3000 K, implying a sound speed of 2×10^{-5}, and

$$\frac{c_s}{aH} \approx 0.003h\,\mathrm{Mpc}^{-1}. \tag{53}$$

The perturbations that we see in the microwave background have wavenumbers $k < 0.05h$ Mpc^{-1}. In the case of galaxy clustering, we may use wavenumbers as high as 0.3 Mpc^{-1}. Thus we see that for the purposes of the CMB, or for setting up the initial conditions for galaxy clustering calculations, we may treat the baryons as cold. (Of course baryon pressure has a large influence on the process of galaxy formation itself.)

The equations. For cold baryons, the perturbation variables are δ_b and $\mathbf{v}_b$, exactly analogous to the CDM variables. The difference is that baryons are acted on by an external force: radiation pressure. The law of conservation of energy-momentum for baryons is then:

$$\nabla_\mu T^{\mu\nu} = F^\nu, \tag{54}$$

where F^ν is the 4-momentum per unit 4-volume delivered to the baryons. This is a 4-vector, and it can be decomposed into $F^{\hat{0}}$ (the power delivered by the radiation per unit volume) and $F^{\hat{\imath}}$ (the 3-force per unit volume).

We investigated the power delivered by the radiation earlier when we considered the Compton effect; it was:

$$F^{\hat{0}} = \frac{4\pi}{15} n_e T_\gamma^4 \sigma_T \frac{T_\gamma - T_m}{m_e} = \frac{3}{2} n \dot{T}_m|_{\mathrm{Compton}}. \tag{55}$$

This power corresponds to the heating of baryons by the CMB and the consequent change in $T^{\hat{0}\hat{0}}$. However as long as we are treating the baryons as cold we can neglect this effect and take

$$F^{\hat{0}} \to 0. \tag{56}$$

Of greater interest is the momentum exchange between the baryons and photons. We found in the lectures on Compton scattering that the net force experienced by an electron was

$$\mathbf{F}_e = -\sigma_T \mathbf{j}'_\gamma, \tag{57}$$

where $\mathbf{j}'_\gamma$ is the radiation momentum density in the baryon rest frame. This is related to the momentum density in the normal frame by a Lorentz transformation,

$$(j'_\gamma)_{\bar{i}} = -T_{(\gamma)\hat{\mu}\hat{\nu}}(\mathbf{e}_{\bar{0}})^{\hat{\mu}}(\mathbf{e}_{\bar{i}})^{\hat{\nu}}, \tag{58}$$

where $\bar{i}$ denotes baryon frame components. The baryon frame basis vectors are

$$(\mathbf{e}_{\bar{0}})^{\hat{\mu}} = (1, v_b^i) \quad \text{and} \quad (\mathbf{e}_{\bar{i}})^{\hat{\nu}} = (v_b^i, \delta_{ij}) \tag{59}$$

to first order in the velocity, so

$$(j'_\gamma)_{\bar{i}} = T_{(\gamma)}^{\hat{0}\hat{i}} - T_{(\gamma)}^{\hat{0}\hat{0}} v_b^i - T_{(\gamma)}^{\hat{i}\hat{j}} v_b^j. \tag{60}$$

Now since v_b is first order, we may replace $T_{(\gamma)}^{\hat{0}\hat{0}}$ and $T_{(\gamma)}^{\hat{i}\hat{j}}$ with their unperturbed values ρ_γ and $\rho_\gamma/3$:

$$(j'_\gamma)_{\bar{i}} = T_{(\gamma)}^{\hat{0}\hat{i}} - \frac{4}{3}\rho_\gamma v_b^i. \tag{61}$$

The component $T_{(\gamma)}^{\hat{0}\hat{i}}$ exists at first order in the photon perturbations and cannot be ignored. It is the momentum density in the normal frame so we can write it as an integral over phase space:

$$T_{(\gamma)}^{\hat{0}\hat{i}} = \int 2\frac{p^2\, dp\, d^2\hat{\mathbf{p}}}{(2\pi)^3} f(p, b\hat{f}p) p\hat{p}^i. \tag{62}$$

Substitute in the equation for f in terms of Θ:

$$\begin{aligned} T_{(\gamma)}^{\hat{0}\hat{i}} &= \frac{1}{4\pi^3}\int d^2\hat{\mathbf{p}}\, \hat{p}^i \int \frac{p^3\, dp}{e^{p/[T_{\gamma 0}(1+\Theta)]} - 1} \\ &= \frac{1}{4\pi^3}\int d^2\hat{\mathbf{p}}\, \hat{p}^i\, \frac{\pi^4}{15} T_{\gamma 0}^4 [1 + 4\Theta(\hat{p}^i)] \\ &= \frac{\rho_\gamma}{\pi}\int d^2\hat{\mathbf{p}}\, \hat{p}^i\, \Theta(\hat{p}^i). \end{aligned} \tag{63}$$

Now each $\hat{p}^i$ is a linear combination of spherical harmonics of order 1:

$$\begin{aligned} \hat{p}^1 &= \sqrt{\frac{2\pi}{3}}[-Y_{11}(\hat{\mathbf{p}}) + Y_{1,-1}(\hat{\mathbf{p}})] \\ \hat{p}^2 &= -\sqrt{\frac{2\pi}{3}}\, i[Y_{11}(\hat{\mathbf{p}}) + Y_{1,-1}(\hat{\mathbf{p}})] \\ \hat{p}^3 &= \sqrt{\frac{4\pi}{3}} Y_{10}(\hat{\mathbf{p}}). \end{aligned} \tag{64}$$

This implies that:

$$T_{(\gamma)}^{\hat{0}\hat{i}} = -4i\rho_\gamma \left(\frac{-\Theta_{11} + \Theta_{1,-1}}{\sqrt{2}}, -i\frac{\Theta_{11} + \Theta_{1,-1}}{\sqrt{2}}, \Theta_{10} \right). \tag{65}$$

The force per unit volume $F^{\hat{i}}$ is then n_e times the force per electron Eq. (57),

$$F^{\hat{i}} = -\frac{4}{3} n_e \sigma_T \rho_\gamma \left(v_b^1 - 3i \frac{\Theta_{11} - \Theta_{1,-1}}{\sqrt{2}}, v_b^2 + 3 \frac{\Theta_{11} + \Theta_{1,-1}}{\sqrt{2}}, v_b^3 + 3i\Theta_{10} \right). \tag{66}$$

The inclusion of this force in the energy-momentum conservation equation gives no change in the baryon density equation:

$$\dot{\delta}_b = -3\dot{D} - \partial_i (B_i + v_b^i). \tag{67}$$

However for the velocity equation, we now have:

$$\dot{v}_b^i = -aHv_b^i - A_{,i} - \frac{4}{3} a n_e \sigma_T \rho_\gamma v_b^i - 4 a n_e \sigma_T \rho_\gamma \left(i \frac{-\Theta_{11} + \Theta_{1,-1}}{\sqrt{2}}, \frac{\Theta_{11} + \Theta_{1,-1}}{\sqrt{2}}, i\Theta_{10} \right). \tag{68}$$

If we do the Fourier-multipole decomposition, we get:

$$\dot{\delta}_b = -3\dot{D} - ik(B_3 + v_b^3) \tag{69}$$

and

$$\dot{v}_b^3 = -aHv_b^3 - ikA - \frac{4}{3} a n_e \sigma_T \rho_\gamma (v_b^i + 3i\Theta_{10}) \tag{70}$$

for the scalars, and

$$\dot{v}_b^{(\pm 1)} = -aHv_b^{(\pm 1)} - \frac{4}{3} a n_e \sigma_T \rho_\gamma [v_b^{(\pm 1)} + 3i\Theta_{1,\pm 1}] \tag{71}$$

for the vectors.

We often define $R \equiv 3\rho_b / 4\rho_\gamma$, which simplifies these equations. Note that $R = 0$ at early times, but by recombination $R \approx 0.4$. This gives us, for the scalars,

$$\begin{aligned} \dot{\delta}_b &= -3\dot{\Phi} - ikv_b; \\ \dot{v}_b &= -aHv_b - ik\Psi + \frac{\dot{\tau}}{R}(v_b + 3i\Theta_1). \end{aligned} \tag{72}$$

For the vectors,

$$\dot{v}_b^{(\pm 1)} = -aHv_b^{(\pm 1)} + \frac{\dot{\tau}}{R}[v_b^{(\pm 1)} + 3i\Theta_{1,\pm 1}]. \tag{73}$$