We now follow the thermal history of the Universe. (c.f. Dodelson 2.4, 3.1, 3.2). The major steps here will be:

- Relativistic plasma ($z \geq 10^{10}$).
- Neutrino decoupling ($z \sim 10^{10}$).
- e^+e^- annihilation ($z \sim 2 \times 10^9$; $T \sim m_e$).
- Big Bang nucleosynthesis ($z \sim 2 \times 10^8$).
- Nonrelativistic plasma.
- Recombination ($z \sim 10^3$).
- Dark ages.
- First stars/galaxies; reionization ($z \sim 10$).
- Cosmological constant ($z < 1$).
- The future.

1 Relativistic plasma

We will concern ourselves first with the equation of state for a relativistic plasma with no chemical potentials, and then find the expansion history. The assumptions of no chemical potential and thermal equilibrium are appropriate at early times, and we'll see when they break down.

Plasma density. Let's consider a gas of bosons and fermions at temperature T. The phase space density of these particles will be:

$$f(\mathbf{q}) = \frac{1}{e^{E(q)/T} \pm 1}, \tag{1}$$

where $\mathbf{q}$ is the momentum,

$$E(q) = \sqrt{m^2 + q^2} \tag{2}$$

is the dispersion relation, and the $+$ sign is for fermions, $-$ for bosons. The total density of these particles is

$$\rho = g \int \frac{d^3\mathbf{q}}{(2\pi)^3} E(\mathbf{p}) f(\mathbf{q}) = \frac{g}{2\pi^2} \int_0^\infty q^2 f(q)\, dq, \tag{3}$$

where g is the degeneracy of the particle ($2s+1$ for massive particles, 2 for photons, 1 for scalars or neutrinos). Can simplify:

$$\rho = \frac{g}{2\pi^2} \int_0^\infty \frac{q^2\sqrt{q^2+m^2}}{\exp[(m^2+q^2)^{1/2}/T] \pm 1} dq. \tag{4}$$

Let's define $x = q/T$, so

$$\rho = \frac{g}{2\pi^2} T^4 \int_0^\infty \frac{x^2\sqrt{x^2 + (m/T)^2}}{e^{\sqrt{(m/T)^2+x^2}} \pm 1} dx. \tag{5}$$

Let's define $\xi = m/T$ and define the integral to be $I_\pm(\xi)$, so that

$$\rho = \frac{g}{2\pi^2} T^4 I_\pm\left(\frac{m}{T}\right). \tag{6}$$

The total density is the sum of this over all of the species present.

A general expression for $I_\pm(\xi)$ is hard but we can do two limiting cases.

Case I: $\xi \to 0$. In this case the integral reduces to

$$I_\pm(0) = \int_0^\infty \frac{x^3}{e^x \pm 1}\, dx. \tag{7}$$

Can use geometric series formula on the denominator:

$$\frac{1}{e^x \pm 1} = \frac{e^{-x}}{1 \pm e^{-x}} = \sum_{j=1}^\infty (\mp 1)^{j-1} e^{-jx}. \tag{8}$$

So:

$$I_\pm(0) = \sum_{j=1}^\infty (\mp 1)^{j-1} \int_0^\infty x^3 e^{-jx}\, dx = \sum_{j=1}^\infty (\mp 1)^{j-1} \frac{3!}{j^4} = 6 \sum_{j=1}^\infty \frac{(\mp 1)^{j-1}}{j^4}. \tag{9}$$

For the bosons we have the + sign and the sum is $\zeta(4) = \pi^4/90$, so

$$I_-(0) = \frac{\pi^4}{15}. \tag{10}$$

For the fermions we have the $-$ sign and

$$\begin{aligned}
I_+(0) &= 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + ...\right) \\
&= 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right) - 12\left(\frac{1}{2^4} + \frac{1}{4^4} + ...\right) \\
&= 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right) - \frac{12}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...\right) \\
&= 6\zeta(4) - \frac{12}{2^4}\zeta(4) \\
&= \left(1 - \frac{12/2^4}{6}\right) I_-(0).
\end{aligned} \tag{11}$$

The factor in parentheses evaluates to 7/8, so

$$I_+(0) = \frac{7}{8} I_-(0). \tag{12}$$

So if all of the species of interest are effectively massless ($m \ll T$) we can write

$$\rho_{\rm tot} = \sum_X \rho_X = \sum_X \frac{g_X}{2\pi^2} T^4 I_\pm(0) \tag{13}$$

Since $I_-(0)/(2\pi^2) = \pi^2/30$,

$$\rho_{\rm tot} = \frac{\pi^2 T^4}{30} \sum_X g_X \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{cases} \tag{14}$$

The summation is often given the letter $g_\star$.

Case II: $\xi \gg 1$, i.e. $m \gg T$. In the absence of a chemical potential we expect very few particles of this species. Let's see how few.

$$I_\pm(\xi) = \int_0^\infty \frac{x^2\sqrt{x^2+\xi^2}}{e^{\sqrt{x^2+\xi^2}} \pm 1} dx. \tag{15}$$

In the denominator the exponential dominates, and increases rapidly if x is comparable to ξ, so only $x \ll \xi$ contributes. Then can replace the square root in the numerator by ξ:

$$I_\pm(\xi) = \int_0^\infty \frac{x^2 \xi}{e^{\sqrt{x^2+\xi^2}}} dx. \tag{16}$$

Let's Taylor expand the square root to the lowest order in x (since $x \ll \xi$):

$$I_\pm(\xi) = \int_0^\infty \frac{x^2\xi}{e^{\xi + x^2/(2\xi)}} dx = \xi e^{-\xi} \int_0^\infty x^2 e^{-x^2/(2\xi)}\, dx. \tag{17}$$

The last integral is Gaussian and evaluates to $\sqrt{\pi/2}\,\xi^{3/2}$. Thus,

$$I_\pm(\xi) = \sqrt{\frac{\pi}{2}}\,\xi^{5/2} e^{-\xi}. \tag{18}$$

From Eq. (10) we can find

$$\frac{I_\pm(\xi)}{I_-(0)} = \frac{15}{2^{1/2}\pi^{7/2}} \xi^{5/2} e^{-\xi}. \tag{19}$$

This is $\ll 1$ so we conclude that a massive species ($m \gg T$) contributes much less to the energy density than the massless ($m \ll T$) species.

If in between, must do numerically. Case I is good to $\sim 10\%$ for $m/T < 0.95$, but for Case II to be valid need $m/T > 35$.

In any case we can define a generalized $g_\star$,

$$g_\star = \sum_X g_X \frac{I_\pm(\xi)}{I_-(0)}, \tag{20}$$

which reduces to the usual equation in the massless case. It is exactly true that

$$\rho = \frac{\pi^2 g_\star T^4}{30}. \tag{21}$$

Plasma pressure and entropy density. For a fully relativistic gas, where the masses are truly zero, we know the pressure is $p = \rho/3$. But we can find the pressure for the general case using the thermodynamic relations. (Also a good review!) Let's define S=total entropy of volume V, s=entropy density. Then from the first law of thermodynamics,

$$dU = T\,dS - p\,dV + \sum_X \mu_X\,dN_X, \tag{22}$$

where μ_X are chemical potentials. But we don't have chemical potentials, so we can drop the last term. Using $U = \rho V$:

$$d(\rho V) = T\,d(sV) - p\,dV \tag{23}$$

or

$$\rho\,dV + V\,d\rho = Ts\,dV + TV\,ds - p\,dV. \tag{24}$$

Now s and ρ depend only on T, not on V so we can write

$$\rho\,dV + V\frac{d\rho}{dT}\,dT = Ts\,dV + TV\,\frac{ds}{dT}\,dT - p\,dV. \tag{25}$$

This is a total differential so equate coefficients of dT and dV:

$$\rho = Ts - p \quad \text{and} \quad V\frac{d\rho}{dT} = TV\,\frac{ds}{dT}. \tag{26}$$

The first equation gives the entropy density in terms of pressure:

$$s = \frac{\rho + p}{T}. \tag{27}$$

The second equation can be divided by V to get:

$$\frac{d\rho}{dT} = T\frac{ds}{dT}. \tag{28}$$

We can find the entropy density by integrating:

$$s(T) = \int_0^T T^{-1}\frac{d\rho(T')}{dT'}dT'. \tag{29}$$

(Recall $s = 0$ at $T = 0$ – 3rd law.)

Can integrate by parts, recalling that $\rho = 0$ at $T = 0$ in the absence of μ. (Set $u = T^{-1}$, $v = \rho$.)

$$s(T) = \left.\frac{\rho(T')}{T'}\right|_0^T + \int_0^T \frac{\rho(T')}{T'^2}dT'. \tag{30}$$

Now $\rho/T \to 0$ as $T \to 0$ since $g_\star$ stays finite:

$$\left.\frac{\rho(T')}{T'}\right|_0^T = \frac{\rho}{T}. \tag{31}$$

Everything can now be expressed in terms of $g_\star$:

$$s(T) = \frac{\pi^2}{30} g_\star T^3 + \frac{\pi^2}{30} \int_0^T g_\star(T') T'^2 dT'. \tag{32}$$

Easier to understand if we write in terms of $y = T'/T$:

$$s(T) = \frac{\pi^2}{30} T^3 \left[g_\star(T) + \int_0^1 g_\star(yT) y^2 \, dy \right]. \tag{33}$$

The pressure can be found from Eq. (27):

$$p(T) = \frac{\pi^2}{30} T^4 \int_0^1 g_\star(yT) y^2 \, dy. \tag{34}$$

And the equation of state is:

$$w(T) = \frac{p(T)}{\rho(T)} = \int_0^1 \frac{g_\star(yT)}{g_\star(T)} y^2 \, dy. \tag{35}$$

If $g_\star$ is a constant, which is true when all species are either massless or irrelevant ($m \gg T$) then $w = 1/3$.

Cases of cosmological interest. Let's consider the function $g_\star$ at early times.

- T ∼few MeV: have photons $g_\gamma = 2$, electrons/positrons ($g_e = 4$), and neutrinos ($g_\nu = 6$ species, one helicity each). Latter two are fermions so

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(6) = \frac{43}{4} = 10.75. \tag{36}$$

- $T \approx 120$ MeV, somewhat below QCD transition: now also have $\mu^+\mu^-$ pairs ($g_\mu = 4$), and pions (bosons, scalars so no spin: $g_\pi = 3$). If the latter were massless, would have

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(6) + \frac{7}{8}(4) + 3 = \frac{69}{4} = 17.25. \tag{37}$$

  (Not perfect since muons/pions have significant mass, and kaons not negligible. Also we've left out e.g. pion self-interactions.)

- At T ∼few hundred MeV, above QCD transiton: now the massless particles are photons ($g_\gamma = 2$), e^+e^- ($g_e = 4$), $\mu^+\mu^-$ ($g_\mu = 4$), neutrinos ($g_\nu = 6$), quarks (3 flavors uds, 3 colors, 2 spin states, and anti-particles, so $g_q = 36$), and gluons (8 colors, 2 polarizations, so $g_g = 16$).

$$g_\star = 2 + \frac{7}{8}(4) + \frac{7}{8}(4) + \frac{7}{8}(6) + \frac{7}{8}(36) + 16 = \frac{247}{4} = 61.75. \tag{38}$$

- At $T \sim 1.5$ GeV, add charm quark ($g_c = 12$) and tau lepton ($g_\tau = 4$). Then $g_\star = 75.75$.
- At $T \sim 5$ GeV, add bottom quark ($g_b = 12$). Then $g_\star = 86.25$.
- At $T > 100$ GeV, add W and Z (3 polarizations, 3 particles, $g_{WZ} = 9$), top $g_t = 12$. Then $g_\star \geq 106.75$ for known particles. Beyond this will need to add Higgs particle(s) and anything else the LHC finds!

Expansion history. The Universe is flat to a good approximation at early times when the horizon is $\ll$ radius of curvature of the Univese. Alternative way to put this is that in the Friedmann equation,

$$\frac{8}{3}\pi G\rho = H^2 + \frac{K}{a^2}, \tag{39}$$

the left hand side $\rho \propto a^{-4}$ for radiation domination (approximately) so this dominates over the curvature term (a^{-2}). So we'll drop the curvature. Using equation for ρ in terms of $g_\star$: ($\rho = \pi^2 g_\star T^4/30$)

$$\frac{4}{45}\pi^3 G g_\star T^4 = H^2. \tag{40}$$

So the Hubble rate is

$$H = \frac{2\pi^{3/2}}{3\sqrt{5}} G^{1/2} g_\star^{1/2} T^2. \tag{41}$$

This is one equation relating the expansion history to temperature, but we need one more equation to close the system for $a(t)$ and $T(t)$ (2 unknowns, need 2 equations). Assuming the Univese remains in thermal equilibrium, which is true prior to neutrino decoupling, the total entropy of the Univese remains fixed (adiabatic expansion). Then entropy density declines as 1/volume:

$$s \propto a^{-3}. \tag{42}$$

We can thus write

$$s = \frac{s_e}{a^3}, \tag{43}$$

where s_e is the extrapolated entropy density of the Universe, i.e. the entropy density today if there were no new sources of entropy. *Warning:* the actual entropy today is $> s_e$ due to non-equilibrium processes, to be discussed later. So we know:

$$\frac{s_e}{a^3} = \frac{\pi^2}{30} T^3 \left[g_\star(T) + \int_0^1 g_\star(yT) y^2 \, dy \right]. \tag{44}$$

Let's consider the case where $g_\star$ is constant over a reasonable range in temperature, so the second integral is $g_\star/3$. Then

$$\frac{s_e}{a^3} = \frac{2\pi^2}{45} g_\star T^3, \tag{45}$$

and

$$T = \left(\frac{45 s_e}{2\pi^2 g_\star}\right)^{1/3} a^{-1}. \tag{46}$$

From Eq. (41):

$$H = \frac{2\pi^{3/2}}{3\sqrt{5}} \left(\frac{90 s_e}{4\pi^2 g_\star}\right)^{2/3} G^{1/2} g_\star^{1/2} a^{-2} = (180\pi)^{1/6} G^{1/2} g_\star^{-1/6} s_e^{2/3} a^{-2}. \tag{47}$$

This is $\dot{a}/a$ so we can re-arrange in terms of an integral for a:

$$a\,da = (180\pi)^{1/6} G^{1/2} g_\star^{-1/6} s_e^{2/3}\,dt. \tag{48}$$

Solve for a by integration:

$$a = 2^{1/2} (180\pi)^{1/12} G^{1/4} g_\star^{-1/12} s_e^{1/3} t^{1/2}. \tag{49}$$

So the expansion history is the usual result ($\propto t^{1/2}$), except that there's a jump every time $g_\star$ changes.

The temperature as a function of time is then

$$T = 2^{-1} 45^{1/4} \pi^{-3/4} G^{-1/4} g_\star^{-1/4} t^{-1/2}. \tag{50}$$

In less clumsy units (i.e. putting in factors of $\hbar$ and c):

$$T = 1.56 g_\star^{-1/4} \sqrt{\frac{1\,\mathrm{s}}{t}}\,\mathrm{MeV}. \tag{51}$$

2 Neutrino decoupling and e^+e^- annihilation

We now want to understand how the relations in the previous section relate to observables, and how we can normalize s_e.

Neutrino decoupling. Neutrinos are kept in equilibrium at early times by reactions such as

$$\nu_x \bar{\nu}_x \leftrightarrow e^+ e^-. \qquad (x = e, \mu, \tau) \tag{52}$$

They decouple at $T \sim$few MeV when $g_\star = 43/4$, after which they redshift as $T \propto 1/a$ (at least ignoring their masses). So let's take the neutrino temperature today, $T_{\nu 0}$, and from Eq. (46):

$$s_e = \frac{43\pi^2}{90} T_{\nu 0}^3. \tag{53}$$

The scale factor as a function of temperature prior to neutrino decoupling is then

$$a = \left(\frac{43}{4 g_\star}\right)^{1/3} \frac{T_{\nu 0}}{T}. \tag{54}$$

e^+e^- **annihilation.** The electrons and positrons annihilate after neutrino decoupling. (Actually they don't completely annihilate since there are a few more e^- than e^+, but this won't concern us yet.) The characteristics of this process are:

- The principal reaction is $e^+e^- \leftrightarrow \gamma\gamma$.
- The neutrinos (almost) don't participate. (A few annihilations go to $e^+e^- \to \nu\bar{\nu}$, affecting results at the $\sim 1\%$ level.)
- The annihilation is fast. The time to reach equilibrium is $\sim \alpha^2/m_e \sim 10^{-18}$ s, much less than the age of the Univese $t \sim$few seconds.

Under these circumstances, the electron-positron-photon plasma adiabatically transitions into a photon-only plasma, i.e. the entropies before and after are the same. Since $g_\star$ for photons is 2 and for $e^+e^-\gamma$ is 11/2, we have

$$\frac{11}{2}T^3a^3\bigg|_{\text{beforeann}} = 2T_\gamma^3a^3\big|_{\text{afterann}}. \tag{55}$$

Now since the neutrinos don't participate, $T_\nu \propto 1/a$ so the temperature on the left hand side applies to neutrinos even after annihilation. Thus, after annihilation,

$$\frac{11}{2}T_\nu^3 = 2T_\gamma^3, \tag{56}$$

or

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \quad \text{(afterannihilation)}. \tag{57}$$

Since $T_{\gamma 0} = 2.73$ K today, the neutrino temperature is:

$$T_{\nu 0} = 1.95\,\text{K}, \tag{58}$$

and we can replace Eq. (54) with something in terms of the photon temperature:

$$a = \left(\frac{43}{11g_\star}\right)^{1/3} \frac{T_{\gamma 0}}{T} \quad \text{(beforeannihilation)}. \tag{59}$$

It's common to define an "effective" $g_{\star,eff}$ such that

$$\rho = \frac{\pi^2}{30}g_{\star,eff}T_\gamma^4, \tag{60}$$

which is equal to $g_\star$ when all species are in equilibrium. This makes e.g. the $T(t)$ relation (Eq. 51) valid. To find $g_{\star,eff}$ after e^+e^- annihilation, we write

$$\rho = \frac{\pi^2}{30}g_{\star,\gamma}T_\gamma^4 + \frac{\pi^2}{30}g_{\star,\nu}T_\nu^4, \tag{61}$$

so

$$g_{\star,eff} = g_{\star,\gamma} + g_{\star,\nu}\left(\frac{T_\nu}{T_\gamma}\right)^4 = 2 + \frac{21}{4}\left(\frac{4}{11}\right)^{4/3} = 3.36. \tag{62}$$

(Actually 3.38 when one takes into account $O(\alpha)$ QED corrections to plasma equation of state and $e^+e^- \to \nu\bar{\nu}$ annihilations.)

Final note: all this assumes the neutrinos are massless, which is a good approximation at high z. Not good today since the neutrino mass is large compared to $T_{\nu 0}$:

- $T_{\nu 0} = 1.95$ K today is 1.7×10^{-4} eV in energy units.
- Solar neutrino mass splitting $\Delta m^2_{21} = (6 \times 10^{-3} \text{ eV})^2$.
- Atmospheric neutrino mass splitting $\Delta m^2_{32} = (0.05 \text{ eV})^2$.

So at least two of the neutrino species are nonrelativistic today. We will discuss the consequences later.