1 Redshift and conformal time

Photon trajectories. Recall from GR that a photon's trajectory is described by $x^\mu(s)$ where s is the *affine parameter*. The photon's momentum is

$$P^\mu = \frac{dx^\mu}{ds}. \tag{1}$$

The photon's energy as seen by an observer with 4-velocity $\mathbf{u}$ is

$$E = -u_\mu P^\mu. \tag{2}$$

In vacuum the photon travels along a geodesic:

$$\frac{dP^\mu}{ds} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta. \tag{3}$$

The photon travels on a null curve, so

$$g_{\mu\nu} P^\mu P^\nu = 0. \tag{4}$$

Redshift. In the last lecture we considered the FRW metric,

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2)]. \tag{5}$$

Let's imagine that we're at the origin $\chi = 0$, looking in some direction (θ, ϕ), at time t_0. A photon is emitted somewhere else in the universe at $(t_i, \chi_i, \theta, \phi)$, initially has energy E_i, and starts traveling toward us. Our job is to determine the energy E_0 that is observed.

By spherical symmetry, a photon traveling radially will stay at constant θ, ϕ. (Literal statement: if $P^\theta = P^\phi = 0$ then $dP^\theta/ds = dP^\phi/ds = 0$.) The requirement of a null trajectory then means

$$-E^2 + a^2(P^\chi)^2 = 0; \tag{6}$$

for inward-going photon,

$$P^\chi = -\frac{E}{a}. \tag{7}$$

So let's get out the Christoffel symbols:

$$\frac{dE}{ds} = -\Gamma^t_{tt} E^2 - \Gamma^t_{t\chi} E \left(-\frac{E}{a}\right) - \Gamma^t_{\chi\chi} \left(-\frac{E}{a}\right)^2. \tag{8}$$

But of these symbols the only nonzero one is

$$\Gamma^t_{\chi\chi} = a\dot{a}, \tag{9}$$

so

$$\frac{dE}{ds} = -\frac{\dot{a}}{a} E^2. \tag{10}$$

We really want dE/dt so use

$$\frac{dt}{ds} = P^t = E, \tag{11}$$

so

$$\dot{E} \equiv \frac{dE}{dt} = \frac{dE/ds}{dt/ds} = -\frac{\dot{a}}{a}E. \tag{12}$$

The solution to this, as expected, is $E \propto a^{-1}$, or in terms of wavelength $\lambda \propto E^{-1} \propto a$.

If a photon was emitted at time t_i then its wavelength observed today must be

$$\lambda_0 = \frac{a(t_0)}{a(t_i)}\lambda_i. \tag{13}$$

It is conventional to define the *redshift* z according to

$$z \equiv \frac{\lambda_0 - \lambda_i}{\lambda_i} = \frac{a(t_0) - a(t_i)}{a(t_i)}. \tag{14}$$

So if we normalize the scale factor today, $a(t_0) = 1$, then

$$a(t_i) = \frac{1}{1+z}. \tag{15}$$

The redshift-time relation $z(t)$ is simply another way to parameterize the expansion of the Universe.

Conformal time. One of the most important quantities in cosmology is the relation between redshift and "distance" – if we see an object at $z = 2$, what is its radial coordinate χ_i where the photon was emitted?

Since the photon's trajectory is null, $u^\mu u_\mu = 0$:

$$-(u^t)^2 + a^2(u^\chi)^2 = 0 \tag{16}$$

for a photon moving radially ($u^\theta = u^\phi = 0$). This means

$$\frac{d\chi}{dt} = \frac{u^\chi}{u^t} = \pm\frac{1}{a}. \tag{17}$$

For a photon moving toward us we take the $-$ solution, so integrate:

$$\chi_0 - \chi_i = -\int_{t_i}^{t_0} \frac{dt}{a}. \tag{18}$$

The radial coordinate today (observed) is $\chi_0 = 0$ so

$$\chi_i = \int_{t_i}^{t_0} \frac{dt}{a}. \tag{19}$$

Let's define the *conformal time* η by

$$\eta(t) = \int \frac{dt}{a}, \tag{20}$$

so

$$\chi_i = \eta_0 - \eta_i \tag{21}$$

where η_0 is the conformal time today and η_i is the conformal time of emission.

We haven't defined the "zero point" of conformal time (constant of integration in Eq. 20). Usually take $\eta = 0$ at the Big Bang, except when studying inflation.

The conformal time η, scale factor a, redshift z, and proper time t are all different possible time coordinates and we are free to choose among them.

When we do perturbation theory it will be useful to write the metric with η instead of t. Since

$$dt = a\, d\eta, \tag{22}$$

we have

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + f(\chi)(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{23}$$

In the special case of a spatially flat universe where $f(\chi) = \chi^2$ the quantity in brackets is the Minkowski metric (in polar coordinates). The factor a^2 is a conformal transformation, hence the name "conformal time."

Horizon. The most distant objects that we can see are the ones whose light left them at the Big Bang. Their radial coordinate χ is

$$\chi = \eta_0. \tag{24}$$

The sphere centered on the observer at this distance is known as the *horizon*. Objects beyond the horizon cannot be seen.

The horizon only exists if the integral $\int dt/a$ is convergent at the Big Bang. (We'll come back to this when we study inflation.)

2 Distance measures

We've already described one measure of distance to a galaxy, χ. This is called the *radial comoving distance*. It is the distance that would be measured at the present epoch by laying down a sequence of rulers end-to-end from us to the galaxy that are at rest with respect to the comoving observers. That is,

$$\chi = \int_{\text{observer}}^{\text{galaxy}} ds, \tag{25}$$

where the path of integration is at constant $t, \theta\phi$ and is at the present, $a = 1$.

The word "comoving" here refers to the fact that the distances are measured with the size of the Universe at the present epoch, not the fact that the rulers are at rest with respect to the comoving observer.

Nobody can measure the radial comoving distance, so we have several other quantities.

- The *angular diameter distance* D_A, which is the distance that goes in the relation between angular diameter $\Delta\theta$ and physical diameter S_{phys}: $\Delta\theta = S/D_A$.

- The *comoving angular diameter distance* r, which is the distance that goes in the relation between angular diameter α and the comoving diameter $S_{\rm com}$ (i.e. the diameter of a stucture today if it expands with the Hubble flow): $\Delta\theta = S_{\rm com}/r$.

- The *luminosity distance* D_L, which is the distance that goes in the relation between flux F of a source (in W/m^2) and luminosity L (in W): $F = L/(4\pi D_L^2)$.

So we need to calculate each of these.

Angular diameter distance. From the metric we can see that the physical diameter subtended by an object is

$$S_{\rm phys} = \int ds = \int \sqrt{ds^2} = \int \sqrt{a^2 f(\chi)\, d\theta^2} = a\sqrt{f(\chi)}\,\Delta\theta. \tag{26}$$

The angular diameter distance is the coefficient of proportionality,

$$D_A = a\sqrt{f(\chi)}. \tag{27}$$

A special case is the spatially flat universe, where $f(\chi) = \chi^2$. Then

$$D_A = a\chi. \tag{28}$$

This distance can be measured if we have an object of known physical size ("standard ruler").

Comoving angular diameter distance. Now instad of using the physical diameter of an object, we will use the comoving diameter, that is the size if it keeps expanding with the Hubble flow. This means that the object expands by a factor of $a_0/a = 1/a$. Thus

$$S_{\rm com} = \frac{S_{\rm phys}}{a} = \sqrt{f(\chi)}\,\Delta\theta. \tag{29}$$

The comoving angular diameter distance is the coefficient of proportionality,

$$r = \sqrt{f(\chi)} = \begin{cases} \chi & K = 0 \\ K^{-1/2}\sin(K^{1/2}\chi) & K > 0 \\ (-K)^{-1/2}\sinh[(-K)^{1/2}\chi] & K < 0 \end{cases} \tag{30}$$

Note that $r = \chi$ for the spatially flat universe.

This distance can be measured if we have an object of known comoving size. Example would be a statistical feature in the distribution of galaxies, which will grow as we go to later times because of Hubble expansion.

For most objects it is easier to measure redshift than distance. If we know z and hence $a = 1/(1+z)$ and one of r or D_A then we can find the other from

$$D_A = a\chi = \frac{\chi}{1+z}. \tag{31}$$

Luminosity distance. A third possible distance arises from the relation of flux to luminosity,

$$F = \frac{L}{4\pi D_L^2}. \tag{32}$$

Need to find constant of proportionality between L and F.

Consider a blackbody of diameter $S_{\rm phys}$ and temperature T. The phase space density of photons emerging from it is

$$f = \frac{1}{e^{E_i/kT} - 1}, \tag{33}$$

where E_i is the energy of the emitted photon. When these photons get to the observer, the energy has declined to

$$E_0 = \frac{E_i}{1+z}. \tag{34}$$

By conservation of phase space density (Liouville's theorem), the phase space density today is

$$f = \frac{1}{e^{E_0(1+z)/kT} - 1}, \tag{35}$$

so the observer sees a blackbody at temperature $T_0 = T/(1+z)$.

Now the luminosity of the object is

$$L = 4\pi \left(\frac{S_{\rm phys}}{2}\right)^2 \sigma T^4. \tag{36}$$

σ = Stefan-Boltzmann constant.

The flux for a blackbody of solid angle

$$\Omega = \pi \left(\frac{\Delta\theta}{2}\right)^2. \tag{37}$$

is

$$F = \frac{\Omega}{\pi} \sigma T_0^4 \tag{38}$$

[Constant out front occurs because the net downward flux in one direction, say the e_3 axis, from a blackbody that fills the sky is σT_0^4. Therefore

$$\sigma T_0^4 = \int_{\rm hemisphere} \frac{dF}{d\Omega} \cos\theta \, d\Omega = \int_0^{2\pi} \int_0^{\pi/2} \frac{dF}{d\Omega} \cos\theta \sin\theta \, d\theta \, d\phi = \pi \frac{dF}{d\Omega}. \tag{39}$$

The $\cos\theta$ comes from the fact that the flux is actually a vector.]

This simplifies to

$$F = \left(\frac{\Delta\theta}{2}\right)^2 \sigma \left(\frac{T}{1+z}\right)^4. \tag{40}$$

Compare to L:

$$\frac{L}{F} = 4\pi\left(\frac{S_{\rm phys}}{\Delta\theta}\right)^2 (1+z)^4. \tag{41}$$

We recognize the ratio $S_{\rm phys}/\Delta\theta$ as the angular diameter distance:

$$\frac{L}{F} = 4\pi D_A^2 (1+z)^4. \tag{42}$$

So the left hand side is $4\pi D_L^2$ and thus

$$D_L = D_A(1+z)^2 = r(1+z). \tag{43}$$

3 Example: Einstein de Sitter Universe

Now that we've done the theory, let's see how these equations play out in the simplest universe, the *Einstein-de Sitter* universe. This is a universe that is spatially flat and consists only of matter ($w = 0$). It is NOT the real universe because it doesn't have a Λ.

Density today is given by the Friedmann equation in terms of the Hubble constant,

$$H_0^2 = \frac{8}{3}\pi G\rho_0, \tag{44}$$

so

$$\rho_0 = \frac{3H_0^2}{8\pi G}. \tag{45}$$

The scale factor as a function of time we solved in the previous lecture:

$$a = \left[\frac{3(1+w)}{2}\right]^{2/3(1+w)} \left(\frac{8}{3}\pi G\rho_0\right)^{1/3(1+w)} t^{2/3(1+w)}. \tag{46}$$

For $w = 0$ and above ρ_0:

$$a = \left(\frac{3}{2}H_0 t\right)^{2/3}. \tag{47}$$

The time today (age of the Universe!) is

$$t_0 = \frac{2}{3H_0}. \tag{48}$$

The Hubble constant as a function of time is

$$H = \frac{2}{3t} = H_0 a^{-3/2}. \tag{49}$$

The time as a function of scale factor is

$$t = \frac{2}{3H_0} a^{3/2}. \tag{50}$$

The conformal time is

$$\eta = \int \frac{dt}{a} = \left(\frac{2}{3H_0}\right)^{2/3} \int \frac{dt}{t^{2/3}} = \left(\frac{2}{3H_0}\right)^{2/3} 3t^{1/3} = \left(\frac{2}{3H_0}\right)^{2/3} 3 \left(\frac{2}{3H_0}\right)^{1/3} a^{1/2} = \frac{2}{H_0} a^{1/2}. \tag{51}$$

The conformal time today (horizon distance!) is

$$\eta_0 = \frac{2}{H_0}. \tag{52}$$

Now let's look at distance-redshift relations. The comoving radial distance is

$$\chi = \eta_0 - \eta = \frac{2}{H_0}(1 - a^{1/2}) = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{53}$$

For a spatially flat universe, r is the same as χ. The angular diameter and luminosity distances differ by a factor of $1+z$:

$$D_A = \frac{2}{H_0(1+z)}\left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{54}$$

and

$$D_L = \frac{2}{H_0}\left(1 + z - \sqrt{1+z}\right). \tag{55}$$

In the limit where $z \ll 1$ can show

$$\chi, r, D_A, D_L \to \frac{z}{H_0}. \tag{56}$$

This relation between the distance and redshift is known as *Hubble's law* and we will show next time that it applies to all nonsingular FRW models.

4 Density parameters, general expressions for the distances

Density parameters. It is often helpful to define the dimensionless *density parameters* that describe the distribution of the Universe's energy in different constituents. To do this let's define the *critical density*:

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G}. \tag{57}$$

From the Friedmann equation we know that this is the density that would make the Universe flat. For each constituent in the Universe, we can define a density parameter:

$$\Omega_X = \frac{\rho_X}{\rho_{\rm crit}}. \tag{58}$$

Here X could be baryons, dark matter, radiation, cosmological constant, etc. The Friedmann equation then says

$$\frac{8}{3}\pi G\rho_{\rm crit}\sum_X \Omega_X = H^2 + \frac{K}{a^2}, \tag{59}$$

or:

$$H^2\sum_X \Omega_X = H^2 + \frac{K}{a^2}. \tag{60}$$

It is common to define a curvature parameter,

$$\Omega_K \equiv -\frac{K}{a^2H^2}. \tag{61}$$

This way the Friedmann equation simply says:

$$\sum_X \Omega_X + \Omega_K = 1. \tag{62}$$

Constraints on curvature are often written in terms of Ω_K. Interpretation:

- For a spatially flat universe, $\Omega_K = 0$.
- For a closed universe, $\Omega_K < 0$ and the radius of curvature is $aR = a/\sqrt{K} = |\Omega_K|^{-1/2}H^{-1}$.
- For an open universe, $\Omega_K > 0$. The imaginary radius of curvature is $i\Omega_K^{1/2}H^{-1}$.

The curvature today ($a = 1$) can be expressed in terms of Ω_K:

$$K = -\Omega_K H_0^2. \tag{63}$$

Warning: Ω_K does not correspond to an actual density, it's just a number introduced to make the Friedmann equation look simple.

Warning 2: In the literature Ω_m, Ω_Λ, etc. usually refer to values today, even though physically they're redshift-dependent.

Expansion history. Given a Hubble constant H_0 and a set of Ωs it's possible to construct the Hubble constant and distances as a function of redshift. We'll consider a universe that has

- Cosmological constant, Λ (density ρ_Λ =constant);
- Curvature, K;
- (Nonrelativistic) matter, m ($\rho_m \propto a^{-3}$);
- Radiation, r ($\rho_r \propto a^{-4}$).

So today the Friedmann equation says

$$\Omega_\Lambda + \Omega_m + \Omega_r + \Omega_K = 1. \tag{64}$$

Let's now go to some previous scale factor a. The total density of the universe back then was

$$\begin{aligned} \rho(a) &= \rho_{\Lambda 0} + \rho_{m0} a^{-3} + \rho_{r0} a^{-4} \\ &= \rho_{\mathrm{crit},0}(\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}). \end{aligned} \tag{65}$$

The Hubble constant at a is:

$$\begin{aligned} H^2 &= \frac{8}{3}\pi G \rho(a) - \frac{K}{a^2} \\ &= \frac{8}{3}\pi G \rho_{\mathrm{crit},0}(\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4}) + \frac{\Omega_K H_0^2}{a^2} \\ &= H_0^2(\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_K a^{-2}). \end{aligned} \tag{66}$$

Let's define the energy function $\mathcal{E}(a)$ by:

$$\mathcal{E}(a) = \sqrt{\Omega_\Lambda + \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_K a^{-2}}. \tag{67}$$

Then the Hubble constant varies with redshift according to

$$H(z) = H_0 \sqrt{\mathcal{E}(z)}. \tag{68}$$

The present age of the universe is:

$$t_0 = \int_{a=0}^{1} dt = \int_0^1 \frac{da}{\dot{a}} = \int_0^1 \frac{da}{aH} = H_0^{-1} \int_0^1 \frac{da}{a\sqrt{\mathcal{E}(a)}}. \tag{69}$$

This integral can also be written in terms of redshift:

$$t_0 = H_0^{-1} \int_0^\infty \frac{dz}{(1+z)\sqrt{\mathcal{E}(z)}}. \tag{70}$$

The age of the universe at some previous time can be written as:

$$t(z) = H_0^{-1} \int_z^\infty \frac{dz'}{(1+z')\sqrt{\mathcal{E}(z')}}. \tag{71}$$

For observers these relations are extremely useful because z is usually observable whereas t, H, etc. are much harder.

Distance measures. The conformal time in one of these models is

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a\dot{a}} = \int \frac{da}{a^2 H} = H_0^{-1} \int \frac{da}{a^2\sqrt{\mathcal{E}(a)}} = -H_0^{-1} \int \frac{dz}{\sqrt{\mathcal{E}(z)}}. \tag{72}$$

So at a particular redshift, and taking the initial condition $\eta = 0$ at $z = \infty$:

$$\eta(z) = H_0^{-1} \int_z^\infty \frac{dz'}{\sqrt{\mathcal{E}(z')}}. \tag{73}$$

The radial comoving distance is:

$$\chi(z) = \eta_0 - \eta(z) = H_0^{-1} \int_0^z \frac{dz'}{\sqrt{\mathcal{E}(z')}}. \tag{74}$$

We can then obtain the other distance measures from

$$r = \begin{cases} \chi & K = 0 \\ K^{-1/2} \sin(K^{1/2}\chi) & K > 0 \\ (-K)^{-1/2} \sinh[(-K)^{1/2}\chi] & K < 0 \end{cases} \tag{75}$$

and

$$D_L = r(1+z); \qquad D_A = \frac{r}{1+z}. \tag{76}$$