Ph217c, Homework 5.
Due Thursday, May 15, 2008.

1. [50%] **Practice with spherical Bessel functions.** For this problem you may use any of the tabulated properties of the Legendre polynomials, but you may not use the properties of the spherical Bessel functions as this would make the problem trivial.

For the purposes of the problem, take the following equation as the definition of the spherical Bessel function:

$$j_l(x) = \frac{(-i)^l}{2} \int_{-1}^{1} e^{ix\mu} P_l(\mu)\, d\mu. \tag{1}$$

(a) Prove the equation

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta). \tag{2}$$

This is the expansion of a plane wave in spherical coordinates.

(b) Using the formula for the Laplacian in spherical coordinates, and the results of (a), prove that j_l satisfies the differential equation:

$$j"_l(x) + \frac{2}{x} j_l'(x) + \left[1 - \frac{l(l+1)}{x^2}\right] j_l(x) = 0. \tag{3}$$

(c) Show that the formula in (b) is equivalent to a Schrödinger equation in one dimension for $xj_l(x)$:

$$\frac{d^2}{dx^2}[xj_l(x)] + \left[1 - \frac{l(l+1)}{x^2}\right] xj_l(x) = 0. \tag{4}$$

In what regions is $j_l(x)$ oscillatory and what are the classically forbidden regions?

(d) Prove that at large x, the spherical Bessel function becomes:

$$j_l(x) \to \frac{1}{x} \sin\left(x - \frac{\pi}{2} l\right). \tag{5}$$

[Hint: deform the contour of integration in Eq. (1) to $-1 \to -1+i\infty \to 1+i\infty \to 1$. Then argue that the integral is dominated by the regions near $\mu \approx -1$ and $\mu \approx 1$.]

(e) Solve the equation in part (c) by the WKB approximation in the oscillatory region. Use the trial form:

$$xj_l(x) = A(x) \sin \alpha(x), \tag{6}$$

with the amplitude A and phase α slowly varying. Ignoring the second derivative $A''(x)$ show that the frequency of oscillation is

$$\alpha'(x) = \sqrt{1 - \frac{l(l+1)}{x^2}} \tag{7}$$

and that the amplitude must satisfy $A \propto \alpha'^{-1/2}$.

(f) Use the limiting result of part (d) to determine the absolute amplitude and phase. Show that if we define β by:

$$x = \sqrt{l(l+1)}\,\sec\beta, \tag{8}$$

that one recovers the solution:

$$x j_l(x) \approx \frac{1}{\sqrt{\sin\beta}} \sin\left[\sqrt{l(l+1)}\,(\tan\beta - \beta) + \phi_0\right]. \tag{9}$$

What is ϕ_0? In the limit of large l, show that $\sqrt{l(l+1)} \to l + \frac{1}{2}$ and $\phi_0 \to \frac{\pi}{4}$, as we assumed in class.

2. [25%] **Spatial curvature.** Consider two cosmological models with $\Omega_m = 0.3$ and $H_0 = 70$ km/s/Mpc. One of them is flat with a cosmological constant, and the other is open with no cosmological constant. Assume their baryon/CDM ratios and initial perturbation spectra are the same.

(a) Evaluate the comoving angular diameter distance to $z = 1000$ in both models.

(b) What is the ratio of the acoustic peak locations l_{peak} in the open model to the locations in the flat+Λ model?

3. [25%] **Photon diffusion length.** Explain qualitatively what each of the following changes would have on the diffusion length k_D^{-1} and hence on the damping tail of the CMB.

(a) An increase in the baryon density $\Omega_b h^2$ with $\Omega_m h^2$ and T_{CMB} held constant.

(b) An increase in the matter density $\Omega_m h^2$ with $\Omega_b h^2$ and T_{CMB} held constant.

(c) Earlier helium recombination.