Ph 217b – Homework 4.
DUE: Thursday, March 13, 2008.

1. [35%] **Alternate derivation of Lyman-α redshifting.** In class we derived the net downward transition rate from emitting Lyman-α photons,

$$\dot{x}_2 = -\frac{H\omega_{\mathrm{Ly}\alpha}^3}{\pi^2 n_{\mathrm{H,tot}}} \left(\frac{x_2}{4x_1} - e^{-\omega_{\mathrm{Ly}\alpha}/T} \right). \tag{1}$$

This problem presents an alternate derivation, also valid in the limit of optically thick lines ($\tau \gg 1$).

(a) Suppose that the decay equation

$$\mathrm{H}(2p) \leftrightarrow \mathrm{H}(1s) + \gamma(\mathrm{Ly}\alpha) \tag{2}$$

comes to equilibrium for photons sufficiently near the Lyman-α frequency, i.e. photons in the Lyman-α line are being rapidly absorbed and emitted. Write down an equation relating the chemical potentials of the ground state $\mu(\mathrm{H}, 1s)$, the excited hydrogen atoms $\mu(\mathrm{H}, n \geq 2)$, and the Lyman-$\alpha$ photons, $\mu(\gamma,\mathrm{Ly}\alpha)$.

(b) Use this equation to derive an equation for the phase space density of Lyman-α photons $f(\omega_{\mathrm{Ly}\alpha})$ in terms of x_1 and x_2. You may assume $x_2 \ll x_1$.

(c) For a given x_1 and x_2, compute the rate at which Lyman-α photons redshift out of the line (in photons/cm^3/s) using your formula for $f(\omega_{\mathrm{Ly}\alpha})$.

(d) Assuming that the photons on the blue side of the Lyman-α line are blackbody photons that have been unaffected by the line, compute the rate (again in photons/cm^3/s) at which photons redshift into the line. You may assume $T \ll \omega_{\mathrm{Ly}\alpha}$.

(e) Argue that the net decay rate $\dot{x}_2$ is related to the difference between your answers to (c) and (d), and thereby derive Eq. (1).

2. [35%] **Stimulated Compton effect.** In class we found that if one considered only spontaneous Compton scattering, the equation for the evolution of the matter temperature exhibited unphysical behavior. The professor said that this problem could be fixed if we included a stimulated Compton heating term,

$$\Gamma_{\mathrm{stim}} = \frac{4\pi^2}{15} n_e T_\gamma^4 \sigma_T \frac{0.041943 T_\gamma}{m_e}. \tag{3}$$

This problem will work through the derivation of this bizarre-looking equation.

(a) In any radiative process, the ratio of the stimulated to the spontaneous transition rate is equal to f, the ambient phase space density of photons in the final state. Prove that in general the stimulated Compton heating rate is given by

$$\Gamma_{\mathrm{stim}} = n_e n_\gamma \sigma_T \frac{\langle \omega^2 f(\omega) \rangle}{m_e}, \tag{4}$$

where the average is taken with equal weighting of each photon.

(b) Evaluate the average value in (a) for a blackbody distribution, and show that $\Gamma_{\rm stim}$ evaluates to Eq. (3).

(c) Stimulated Compton scattering should also apply to the Compton cooling discussed in class. Explain why (at least to the order to which we worked in class) $\Lambda_{\rm stim} = 0$.

3. [15%] **Compton cooling at low redshift.** Suppose a cloud of fully ionized gas is much hotter than the CMB. Using the formulas derived in class, show that at $z = 0$ the time required for it to Compton-cool is longer than the age of the Universe. But what about at $z = 10$?

4. [15%] **The spectral distortion.** Suppose that about half of the hydrogen atom recombinations produce Lyman-α photons.

(a) What is the wavelength of these photons today?

(b) What is the energy flux of these photons (in erg/cm^2/s/steradian, or W/m^2/steradian) that would be seen by a present-day detector?

(c) The cosmic infrared background consists of radiation emitted over the lifetime of the Universe by interstellar dust grains in distant galaxies. The flux $dI/d\ln\lambda$ was estimated by Schlegel et al. (1997) to be 32 ± 13 nW/m^2/sr at 140 μm, and 17 ± 4 nW/m^2/sr at 240 μm. Do the above-mentioned Lyman-α photons make a significant contribution to the infrared background?