Ph217c, Homework 6.
Due Thursday, May 29, 2008.

1. [50%] **Properties of tensor spherical harmonics.** Consider the tensor spherical harmonics $Y^E_{lm\ ab}(\theta,\phi)$ and $Y^B_{lm\ ab}(\theta,\phi)$ discussed in class:

$$\begin{aligned} Y^E_{lm\ ab}(\theta,\phi) &= -\frac{2}{\sqrt{(l-1)l(l+1)(l+2)}}\left(D_aD_b - \frac{1}{2}g_{ab}D^2\right)Y_{lm}(\theta,\phi), \\ Y^B_{lm\ ab}(\theta,\phi) &= -\frac{1}{\sqrt{(l-1)l(l+1)(l+2)}}\left(\epsilon_{bc}D_aD_c + \epsilon_{ac}D_bD_c\right)Y_{lm}(\theta,\phi), \end{aligned} \tag{1}$$

where D_a is the covariant derivative on the unit sphere, g_{ab} is the metric tensor on the unit sphere,

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{2}$$

ϵ_{ab} is the Levi-Cevita tensor,

$$\epsilon_{ab} = \frac{1}{\sin\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

and D^2 is the Laplacian, $D^2 = g^{ab}D_aD_b$.

(a) Prove that $Y^E_{lm\ ab}(\theta,\phi)$ and $Y^B_{lm\ ab}(\theta,\phi)$ are symmetric and traceless.

(b) Find the Riemann tensor $R^a{}_{bcd}$ for the unit sphere, and write it in terms of g_{ab} and δ^a_b.

(c) Prove the orthonormality relation:

$$\int Y^{E*}_{lm\ ab}(\hat{\mathbf{p}})Y^{E\ ab}_{l'm'}(\hat{\mathbf{p}})\,d^2\hat{\mathbf{p}} = 2\delta_{ll'}\delta_{mm'}. \tag{4}$$

Hint: expand this integral using Eq. (1), and use integration by parts to move all derivatives onto $Y_{l'm'}$. Then use the commutation rule for covariant derivatives, $[D_b, D_c]A_a = R^d{}_{acb}A_d$, to eliminate the tensor indices and recover an integral involving scalar spherical harmonics.

(d) Using similar techniques to (c), prove the analogous orthonormality relations:

$$\int Y^{B*}_{lm\ ab}(\hat{\mathbf{p}})Y^{B\ ab}_{l'm'}(\hat{\mathbf{p}})\,d^2\hat{\mathbf{p}} = 2\delta_{ll'}\delta_{mm'} \tag{5}$$

and

$$\int Y^{E*}_{lm\ ab}(\hat{\mathbf{p}})Y^{B\ ab}_{l'm'}(\hat{\mathbf{p}})\,d^2\hat{\mathbf{p}} = 0. \tag{6}$$

(e) Explain why the tensor spherical harmonics only exist for $l \geq 2$.

(f) Write down the explicit expressions for the $l = 2$ tensor spherical harmonics. You should write the tensor components in the orthonormal basis $\{\mathbf{e}_{\hat\theta}, \mathbf{e}_{\hat\phi}\}$.

2. [20%] **Polarization power spectrum.** Define the RMS polarization of the CMB in a particular direction as:

$$P_{\rm rms} = \sqrt{\langle Q^2(\hat{\mathbf{p}}) + U^2(\hat{\mathbf{p}}) \rangle}. \tag{7}$$

Show that

$$P_{\rm rms}^2 = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} (C_l^{EE} + C_l^{BB}). \tag{8}$$

3. [30%] **Cluster polarization.** Suppose that we lived at the center of a galaxy cluster. The intracluster gas is ionized; assume for the purposes of this problem that it is a sphere of electron density $n_e = 10^{-3}$ cm^{-3} and radius $R = 10^{24}$ cm. (Real clusters have a density that decreases as one goes outward but we won't invoke such complications here.)

(a) What is the Thomson optical depth that we would see?

(b) Electron scattering within the cluster will generate CMB polarization. Estimate the RMS polarization that we would see from the cluster center. What multipole l does this polarization correspond to? Is it of the E or B type?

(c) What would the cluster look like to an outside observer with a microwave polarization telescope?