Interstellar Hydrodynamics

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1. Introduction

Thus far we have considered several of the major phases of the ISM and their chemical and thermal state. However, we have not yet considered the motion of the ISM. On the largest scales, the gas in the Milky Way's disk is orbiting on roughly circular orbits in the Galactic potential, and it undergoes some radial motion due to the passage of spiral arms. On smaller scales, the disk has a thickness determined by its sources of pressure that try to puff up the disk, balanced by the force of gravity that acts to compress it. On even smaller scales, the ISM undergoes velocity flows associated with gravity, supernova explosions, and responses to variations in pressure due to heating. Finally, the smallest-scale velocity flows are due to **turbulence**: large-scale velocity flows tend to be unstable and cascade to smaller scales (a familiar phenomenon from everyday life).

A new aspect of ISM dynamics is the role of the magnetic field. The ISM – even its "neutral" component – contains free charges and hence is an electric conductor. This means that motions in a magnetized ISM lead to large-scale currents, which then are affected by $\mathbf{J}\times\mathbf{B}$ forces. The dynamics of conducting magnetized fluids is called **magnetohydrodynamics** (MHD) and plays a key role in the study of the ISM (and of other systems, e.g. accretion disks and jets, stellar and planetary magnetism, and laboratory and space plasmas).

We will consider only nonrelativistic hydrodynamics and MHD, as these are relevant to the ISM, but in other applications (e.g. near black holes) their relativistic analogues are important.

References:

- Osterbrock & Ferland, Ch. 6 (first few sections)

- MHD waves are described in some advanced E&M textbooks (e.g. Jackson §7.7) or any book on plasma physics.

2. The Basic Equations

A. HYDRODYNAMICS

The basic equations of hydrodynamics are the conservation of mass, momentum, and energy; and the basic variables are the density ρ, velocity $\mathbf{v}$, and temperature T. The conservation of mass is the simplest: the flux of mass is $\rho\mathbf{v}$ (units: g cm^{-2} s^{-1}), so the mass density must change in accordance with:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v}).$$

The partial derivative indicates that the derivative is to be taken at constant position **x**. This equation is often written in an alternative form, using the **convective derivative**:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla.$$

For any scalar quantity Φ, an observer sitting at a fixed point **x** measures a time derivative $\partial\Phi/\partial t$, whereas an observer moving with the flow of the fluid measures $d\Phi/dt$. The convective derivative of the density is:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = -\nabla\cdot(\rho\mathbf{v}) + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v}.$$

Our second basic equation is the conservation of momentum. The flux of momentum [units: (g cm s^{-1}) cm^{-2} s^{-1} = g cm^{-1} s^{-2}; same units as energy density or pressure] **T** is actually a tensor quantity: that is, there is a flux of the i component of momentum (i=x,y,z) in the j direction. It can thus be represented by a 3×3 matrix, and we call it the **stress tensor**. An additional feature of momentum is that unlike mass, it can be added by external forces: if there is a force density (dyne cm^{-3}), e.g. the $\rho\mathbf{g}$ force due to gravity, then the momentum density contains an additional time derivative due to this external force. Since the momentum density is ρv_i, we have:

$$\frac{\partial}{\partial t}(\rho v_i) = F_i - \sum_j \frac{\partial T_{ij}}{\partial x_j}.$$

The momentum flux contains at least two components: that due to the internal pressure of the gas, and that due to its motion. The motion contributes a momentum flux of the momentum density ρv_i times the velocity at which it is transported v_j, or $\rho v_i v_j$. The internal pressure for an ideal gas only transports x momentum in the x direction, etc. due to the spherical symmetry of the Maxwellian distribution. This transport rate is $P\delta_{ij}$. Therefore, we have a stress tensor:

$$T_{ij} = \rho v_i v_j + P\delta_{ij}.$$

In practice, there may be additional contributions to the stress-energy due to **viscosity** – the phenomenon that a gas in an inhomogeneous flow (i.e. where **v** depends on **x**) is not quite describable by a Maxwellian distribution. For flows where the velocity varies on scales long compared to the mean free path, we may usually treat viscosity as contributing a small contribution to T_{ij} that depends on the gradient of the velocity. Symmetry considerations, combined with the fact that T_{ij} must be symmetric (a general law of physics, but in the case of a gas this is easily derivable by considering that an individual molecule with velocity **u** has a contribution proportional to $u_i u_j$) restrict this contribution to the form:

$$T_{ij} = \rho v_i v_j + P\delta_{ij} - \frac{1}{3}\kappa\delta_{ij}(\nabla\cdot\mathbf{v}) - \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Here κ is called the **bulk viscosity coefficient** and η the **shear viscosity coefficient**. They are generally both positive for stable media.

The equation for the velocity is then:

$$\frac{\partial}{\partial t}(\rho v_i) = F_i - \sum_j \frac{\partial}{\partial x_j}\left[\rho v_i v_j + P\delta_{ij} - \frac{1}{3}\kappa\delta_{ij}(\nabla\cdot\mathbf{v}) - \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right].$$

A case of common interest is where the viscosity is negligible, in which case the above equation simplifies to:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = \mathbf{F} - \mathbf{v}\cdot\nabla(\rho\mathbf{v}) - \rho\mathbf{v}\nabla\cdot\mathbf{v} - \nabla P.$$

Moving the **v**•∇ term to the left-hand side gives a convective derivative:

$$\frac{d}{dt}(\rho\mathbf{v}) = \mathbf{F} - \rho\mathbf{v}\nabla\cdot\mathbf{v} - \nabla P.$$

Using the product rule, and the convective derivative formula for ρ, we can simplify this to:

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{F} - \nabla P.$$

The above equations are called the **Navier-Stokes equations** and describe gas flow in many situations (except shocks).

In order to close the above equations, one needs one more equation: the conservation of energy. The energy density can be represented as the sum of kinetic energy density $\rho v^2/2$, and internal energy ρe, where e is the internal energy per unit mass and depends on the equation of state. For ideal monatomic gases (our main concern), $e = 3kT/2\mu$ (where μ is the mean molecular weight). We also need the energy flux, which has several parts. One is simple advection of the internal energy, i.e. $\rho e\mathbf{v}$. A second is the application of pressure to a moving surface, $-P\mathbf{v}$. A third is thermal conduction, $-\kappa_{\rm th}\nabla T$, where $\kappa_{\rm th}$ is the **thermal conductivity**. Finally, we need the net sources of energy, which are the net heating Q (erg cm^{-3} s^{-1}), including viscous heating; and the work done by external forces $\mathbf{F}\bullet\mathbf{v}$. We then have:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e\right) = -\nabla\cdot\left(\frac{1}{2}\rho v^2\mathbf{v} + \rho e\mathbf{v} - P\mathbf{v} - \kappa_{\rm th}\nabla T\right) + Q + \mathbf{F}\cdot\mathbf{v}.$$

We will focus here on the idealized case where thermal conduction is negligible.

Writing this in terms of the convective derivative gives:

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\rho v^2 + \rho e\right) &= -\nabla\cdot\left(\frac{1}{2}\rho v^2\mathbf{v} + \rho e\mathbf{v} - P\mathbf{v}\right) + \mathbf{v}\cdot\nabla\left(\frac{1}{2}\rho v^2 + \rho e\right) + Q + \mathbf{F}\cdot\mathbf{v} \\ &= -\left(\frac{1}{2}\rho v^2 + \rho e\right)\nabla\cdot\mathbf{v} + \nabla\cdot(P\mathbf{v}) + Q + \mathbf{F}\cdot\mathbf{v}.\end{aligned}$$

We may now extract ρ from the left-hand side using the product rule, and cancel the $\nabla\bullet\mathbf{v}$ term on the right:

$$\rho\frac{d}{dt}\left(\frac{1}{2}v^2 + e\right) = \nabla\cdot(P\mathbf{v}) + Q + \mathbf{F}\cdot\mathbf{v}.$$

Finally, we explicitly take the convective derivative of $v^2/2$:

$$\frac{d}{dt}\frac{v^2}{2} = \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \frac{1}{\rho}\left(\mathbf{v}\cdot\mathbf{F} - \mathbf{v}\cdot\nabla P\right),$$

which allows us to simplify the energy equation to:

$$\rho\frac{de}{dt} = P\nabla\cdot\mathbf{v} + Q.$$

If we replace $\nabla\bullet\mathbf{v}$ with $-(d\rho/dt)/\rho$, this becomes

$$\rho \frac{de}{dt} = -\rho P \frac{d}{dt}\left(\frac{1}{\rho}\right) + Q.$$

It is convenient to define the **specific enthalpy** h by $h = e + P/\rho$. For an ideal monatomic gas, $h = 5kT/2\mu$. We may then write:

$$\rho \frac{dh}{dt} = \rho \frac{de}{dt} + \rho P \frac{d}{dt}\left(\frac{1}{\rho}\right) + \frac{dP}{dt} = Q + \frac{dP}{dt}.$$

It may also be valuable to write the above equations in terms of the **stagnation enthalpy** $h_0 = h + v^2/2$:

$$\begin{aligned} \rho \frac{dh_0}{dt} &= \rho \frac{d}{dt}\left(\frac{1}{2}v^2 + e\right) + \rho \frac{d}{dt}\frac{P}{\rho} \\ &= \nabla \cdot (P\mathbf{v}) + Q + \mathbf{v}\cdot\mathbf{F} + \frac{dP}{dt} - P\nabla\cdot\mathbf{v} \\ &= Q + \frac{\partial P}{\partial t} + \mathbf{v}\cdot\mathbf{F}. \end{aligned}$$

The stagnation enthalpy is useful because in time-steady situations and in the absence of dissipation, external forces, and sources of energy, it is conserved along a trajectory.[1] We will use this fact to our advantage in the study of shocks.[2]

The above equations can only be closed in general by an **equation of state**, which is a function that relates the enthalpy h, the density ρ, the temperature T, and the pressure P (and, in the presence of dissipation, the transport coefficients κ, η, and κ_{th}) to each other such that only two variables (e.g. ρ and T) are required to describe all of these. The monatomic ideal gas equation of state:

$$P = \frac{k\rho T}{\mu}, \quad h = \frac{5kT}{2\mu},$$

is applicable for most of our purposes. Furthermore, we require a formula for Q if the gas is being followed over a timescale comparable to the heating or cooling time (from photoionization or photoelectric heating, or radiative cooling).

B. THE HEATING TERM

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[1] If the external force is conservative, $\mathbf{F} = -\rho\nabla\Phi$, then $h_0 + \Phi$ is conserved along a trajectory.

[2] This equation is still valid at shocks because it simply derives from conservation of energy. The flow velocity is formally not defined within the shock, so the equation for $d(v^2/2)/dt$ is not valid; this is why we cannot use the h equation in a shock.

There are two very important limits in the above discussion. One is the limit of no significant heating or cooling, in which case we may set $Q = 0$. The energy equation then says $\rho\, dh/dt = dP/dt$. In this case, with the exception of behavior at shocks (where one cannot ignore sharp changes in the velocity field in the above derivation!), we have:

$$\frac{5k\rho T}{2\mu}\frac{d\ln T}{dt} = \frac{k\rho T}{\mu}\left(\frac{d\ln\rho}{dt} + \frac{d\ln T}{dt}\right).$$

This implies that:

$$\frac{d\ln T}{dt} = \frac{2}{3}\frac{d\ln\rho}{dt}.$$

Thus, in this case we find that the temperature and pressure of the gas obey $T = K\rho^{2/3}$. The coefficient of proportionality is generally different for different parcels of gas, but in many problems (e.g. initially homogeneous media with some T_0 and ρ_0), the coefficient of proportionality can be taken as fixed. The pressure, which appears in the momentum equation, is then:

$$P = P_0\left(\frac{\rho}{\rho_0}\right)^{5/3}.$$

This will be recognized as the standard adiabatic relation. For small perturbations in the pressure, one may then write:

$$\delta P = c_s^2\delta\rho, \quad c_s^2 = \frac{5kT}{3\mu},$$

where c_s is called the **adiabatic sound speed** (so far just a name!).

The opposite situation occurs in a medium where the thermal equilibrium timescale is short compared to the timescale associated with the flow. In this case, if the medium is thermally stable, Q adjusts itself so that the medium stays on the equilibrium curve in the (ρ,T) plane. The thermally stable curve is usually close to T = constant (since cooling is exponentially sensitive to T), so it is often a good approximation to set T = constant. In this case, we have for small perturbations

$$\delta P = c_s^2\delta\rho, \quad c_s^2 = \frac{kT}{\mu},$$

where this c_s is now the **isothermal sound speed**.

C. MAGNETOHYDRODYNAMICS

We are now ready to include magnetic fields in the above analysis. The force per unit volume on a parcel of fluid is now $\mathbf{F} = \mathbf{J} \times \mathbf{B} / c$. Using Ampère's law, and working in the nonrelativistic limit where the displacement current is small, we may then write:

$$\mathbf{F} = \frac{1}{c}\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}.$$

The momentum equation then reads:

$$\rho \frac{d\mathbf{v}}{dt} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla P.$$

It is of conceptual importance that this force can be written as the divergence of a stress tensor. Vector identities show that

$$\mathbf{F} = -\frac{1}{8\pi}\nabla(B^2) + \frac{1}{4\pi}\mathbf{B} \cdot \nabla\mathbf{B}.$$

Then using the fact that $\mathbf{B}$ is divergenceless, we may write

$$F_i = -\sum_j \frac{\partial T_{ij}^{(B)}}{\partial x_j}, \qquad T_{ij}^{(B)} \equiv \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi}.$$

Here $\mathbf{T}^{(B)}$ can be thought of as a magnetic stress tensor (in fact, in the general theory of electrodynamics it is). The above calculation suggests that in directions along the magnetic field lines, the stress is negative (i.e. field lines pull like tensioned strings), whereas in the perpendicular-to-field directions the stress is positive (field lines push against each other). Many qualitative results of MHD can be conceptualized with this picture.

In order to close the system of equations, we need a formula for the behavior of $\mathbf{B}$. This is given by Ohm's law, which relates the current density $\mathbf{J}$ to the electric field:

$$\sigma^{-1}\mathbf{J} = \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c},$$

where σ is the conductivity. The term on the right-hand side involving the magnetic field takes note of the fact that a moving observer sees a slightly different electric field than a stationary observer. Taking the curl of the above equation gives:

$$\nabla \times (\sigma^{-1}\mathbf{J}) = \nabla \times \mathbf{E} + \frac{\nabla \times (\mathbf{v} \times \mathbf{B})}{c}$$
$$= -c^{-1}\frac{\partial \mathbf{B}}{\partial t} + c^{-1}\left[\mathbf{v}(\nabla \cdot \mathbf{B}) + \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{B}\right].$$

The divergencelessness of **B** eliminates one term on the right hand side. Also, Ampère's law allows us to simplify the left-hand side to:

$$\frac{c}{4\pi}\nabla \times (\sigma^{-1}\nabla \times \mathbf{B}) = -c^{-1}\frac{\partial \mathbf{B}}{\partial t} + c^{-1}\left[\mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{B}\right].$$

If σ = constant (which is not generally true, but is often a reasonable approximation, especially in situations where we may set $\sigma^{-1} \rightarrow 0$), then we find:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla \mathbf{B} - \frac{c^2}{4\pi\sigma}\nabla \times (\nabla \times \mathbf{B}).$$

Defining the **magnetic diffusivity** $\chi = c^2 / 4\pi\sigma$, we simplify this to:

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) + \chi \nabla^2 \mathbf{B}.$$

Conceptually, the first term on the right-hand side reflects the shearing of magnetic field lines by velocity gradients; the second term represents their compression or expansion; and the third represents diffusion. Typical diffusion coefficients for an ionized plasma are $\sim 10^6$ cm^2/s, so a field line will diffuse << 1 pc in the age of the Universe. Thus in many situations it is valid to neglect the diffusion (we will see exceptions later).

If χ can be neglected, then it can be shown that the magnetic flux through a loop that moves with the fluid is conserved. We may see this explicitly by using Stokes's theorem for the flux:

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{n} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{x}.$$

Then:

$$\frac{d\Psi}{dt} = \frac{d}{dt}\oint_{\partial S} \mathbf{A} \cdot d\mathbf{x} = \oint_{\partial S}\left[\frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{x} + (\mathbf{v} \cdot \nabla \mathbf{A}) \cdot d\mathbf{x} + (d\mathbf{x} \cdot \nabla \mathbf{v}) \cdot \mathbf{A}\right],$$

where the last term takes account of the fact that the differential distance $d\mathbf{x}$ is changing as the loop deforms. The last term may be written using the product rule as:

$$\frac{d\Psi}{dt} = \oint_{\partial S} \left[\frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{x} + (\mathbf{v} \cdot \nabla \mathbf{A}) \cdot d\mathbf{x} + d\mathbf{x} \cdot \nabla(\mathbf{v} \cdot \mathbf{A}) - (d\mathbf{x} \cdot \nabla \mathbf{A}) \cdot \mathbf{v} \right].$$

The term involving $\nabla(\mathbf{v} \bullet \mathbf{A})$ is a total derivative and integrates to zero, so we may drop it. The second and fourth terms may be combined by vector identities to give:

$$\frac{d\Psi}{dt} = \oint_{\partial S} \left[\frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{x} + (\nabla \times \mathbf{A}) \cdot (\mathbf{v} \times d\mathbf{x}) \right].$$

The curl of **A** is **B**, and cyclic permutation of the triple product then gives:

$$\frac{d\Psi}{dt} = \oint_{\partial S} \left[\frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{x} + (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{x} \right].$$

We may now use Stokes's theorem to go back to integrals over the surface *S*:

$$\begin{aligned} \frac{d\Psi}{dt} &= \int_S \left[\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) \right] \cdot d\hat{\mathbf{n}} \\ &= \int_S \left[\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{v} \right] \cdot d\hat{\mathbf{n}}. \end{aligned}$$

Our equation for the evolution of **B** shows that the integrand is zero. Thus we see that magnetic field lines are trapped to the plasma: if the fluid moves, so does the field and vice versa. This is of course unsurprising for an idealized perfect conductor.

3. Simple Solutions

We are now ready to consider some simple solutions to the equations of hydrodynamics and MHD. We will focus on two examples of direct relevance to the ISM: the propagation of waves, and the plane-parallel disk (a model for the gaseous disk of our Galaxy). A third example – shocks – will be covered in the next lecture.

A. WAVES IN HYDRODYNAMICS

Our first consideration is waves in hydrodynamics. We will assume a uniform background medium and only consider linear perturbations. We will also assume for simplicity that one of the two limiting cases (adiabatic or isothermal) applies to the waves. The background density is taken to be ρ_0, the appropriate sound speed (either adiabatic or isothermal) is c_s, and the unperturbed medium is at rest, $\mathbf{v}=0$. This is a simple example, but it serves as a warm-up for what comes next.

The basic setup for a wave problem is to write the perturbed quantities as complex exponentials,

$$\delta\rho \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}.$$

Then a spatial gradient acting on a perturbation undergoes the replacement $\nabla \to i\mathbf{k}$ and the time derivative is $\partial/\partial t \to -i\omega$. Since the background is stationary, convective derivatives can to linear order be replaced with partial derivatives. In more general analyses where the background is moving, this replacement would not necessarily be valid.

The density and velocity equations then become

$$-i\omega\delta\rho = -i\mathbf{k}\cdot\rho_0\mathbf{v},$$
$$-i\omega\rho_0\mathbf{v} = -i\mathbf{k}\delta P = -i\mathbf{k}c_s^2\delta\rho.$$

The first equation can be used to eliminate the density perturbation,

$$\omega\delta\rho = \rho_0\mathbf{k}\cdot\mathbf{v}.$$

Then the second equation reduces to:

$$\omega^2\mathbf{v} = c_s^2(\mathbf{k}\cdot\mathbf{v})\mathbf{k},$$

which has two types of solutions. If $\mathbf{k}\|\mathbf{v}$, then we have a compressional wave ($\delta\rho\neq 0$) with dispersion relation $\omega=c_s k$. This is the conventional acoustic wave. If $\mathbf{k}\perp\mathbf{v}$, then we find a zero mode, i.e. a non-oscillatory mode with $\omega=0$ and $\delta\rho=0$. This mode corresponds to extremely slow divergence-free motions. It represents the degrees of freedom of incompressible hydrodynamics, and if excited its ultimate fate is that the convective derivative terms eventually become important and result in complicated (often turbulent) motions. We will discuss these later when we describe interstellar turbulence.

B. WAVES IN MAGNETOHYDRODYNAMICS

We now consider a somewhat more complicated problem: we suppose that the above fluid is conducting and lives in a uniform background magnetic field $\mathbf{B}_0$. The equations of motion will be used to discover the possible wave motions in MHD.

The density equation is of course not modified:

$$-i\omega\delta\rho = -i\mathbf{k}\cdot\rho_0\mathbf{v}.$$

The velocity equation now contains not just the pressure force, but an additional term. We note that $\nabla\times\mathbf{B} = i\mathbf{k}\times\delta\mathbf{B}$ since the background field is uniform; then:

$$-i\omega\rho_0\mathbf{v} = \frac{1}{4\pi}(i\mathbf{k}\times\delta\mathbf{B})\times\mathbf{B}_0 - i\mathbf{k}c_s^2\delta\rho.$$

Finally, we need the magnetic field equation,

$$-i\omega\delta\mathbf{B} = i(\mathbf{B}_0 \cdot \mathbf{k})\mathbf{v} - i\mathbf{B}_0(\mathbf{k} \cdot \mathbf{v}).$$

The first and last equations can be used to eliminate δρ and δ**B.** This gives:

$$-i\omega\rho_0\mathbf{v} = \frac{1}{4\pi\omega}\{i\mathbf{k} \times [-(\mathbf{B}_0 \cdot \mathbf{k})\mathbf{v} + \mathbf{B}_0(\mathbf{k} \cdot \mathbf{v})]\} \times \mathbf{B}_0 - i\mathbf{k}c_s^2\rho_0\frac{\mathbf{k} \cdot \mathbf{v}}{\omega}.$$

In order to simplify this equation further, we define a new velocity, called the **Alfvén velocity** $\mathbf{v}_A$:

$$\mathbf{v}_A = \frac{\mathbf{B}_0}{\sqrt{4\pi\rho_0}}.$$

This can be thought of as a vector since the background field selects a pure direction. Then the equation for the velocity becomes:

$$\omega^2\mathbf{v} = \{\mathbf{k} \times [(\mathbf{v}_A \cdot \mathbf{k})\mathbf{v} - \mathbf{v}_A(\mathbf{k} \cdot \mathbf{v})]\} \times \mathbf{v}_A + c_s^2\mathbf{k}(\mathbf{k} \cdot \mathbf{v}).$$

This equation simplifies using vector identities:

$$\omega^2\mathbf{v} = (\mathbf{v}_A \cdot \mathbf{k})^2\mathbf{v} - (\mathbf{k} \cdot \mathbf{v})(\mathbf{k} \cdot \mathbf{v}_A)\mathbf{v}_A - (\mathbf{k} \cdot \mathbf{v}_A)(\mathbf{v} \cdot \mathbf{v}_A)\mathbf{k} + v_A^2(\mathbf{k} \cdot \mathbf{v})\mathbf{k} + c_s^2\mathbf{k}(\mathbf{k} \cdot \mathbf{v}).$$

To study the solutions, we will place the background field in the z-direction, and **k** in the xz-plane without loss of generality. Then one would guess based on symmetry principles that the waves with motion in the y-direction would decouple from those with motion in the xz-plane. This is indeed the case. If we take **v** to be in the y-direction, the above equation is trivial in the x and z components; the y component is:

$$\omega^2 v_y = (\mathbf{v}_A \cdot \mathbf{k})^2 v_y.$$

We see that this wave has a dispersion relation $\omega = v_A|k_z|$. The waves thus have a group velocity of $\pm\mathbf{v}_A$, and thus represent excitations that propagate up and down the field lines at fixed velocity. Such waves are called **Alfvén waves** and have no hydrodynamic analogue. They are always noncompressional. One can see from the δ**B** equation that δ**B** is perpendicular to $\mathbf{B}_0$, i.e. the magnetic field does not change its strength, only its direction.

We now consider the wave modes with velocities in the xz-plane. We write θ as the angle between **k** and $\mathbf{B}_0$. Then our equation has two nontrivial components in the x and z directions:

$$\omega^2 \begin{pmatrix} v_x \\ v_z \end{pmatrix} = k^2 \begin{pmatrix} v_A^2 + c_s^2 \sin^2\theta & c_s^2 \sin\theta\cos\theta \\ c_s^2 \sin\theta\cos\theta & c_s^2 \cos^2\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_z \end{pmatrix}.$$

The wave solutions correspond to the eigenvectors of the matrix and their velocities are associated with the eigenvalues. The eigenvalue problem reduces to:

$$\omega^4 - (v_A^2 + c_s^2)k^2\omega^2 + v_A^2 c_s^2 k^4 \cos^2\theta = 0.$$

The solutions are:

$$\omega^2 = \frac{k^2}{2}\left[v_A^2 + c_s^2 \pm \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2\theta}\right].$$

The upper and lower solutions are called the **fast magnetosonic wave** and **slow magnetosonic wave**, respectively. Note that the dispersion relation is invariant if we swap v_A and c_s, although the eigenvector (direction of **v**) is changed.

In the limit of a weak magnetic field $v_A \ll c_s$, we can see that the fast mode becomes an acoustic wave: the dispersion relation is $\omega = c_s k$. The slow mode in this limit has the dispersion relation $\omega = v_A|k|$, i.e. the same as for the Alfvén wave. In this limit, both the Alfvén wave and the slow wave are noncompressional motions of the fluid that twist the magnetic field lines; magnetic tension then provides the restoring force for the waves (in some ways, they are just like waves on a 1D string – c.f. Phys 12a).

In the limit of a strong magnetic field $v_A \gg c_s$, the fast wave propagates at velocity v_A – but its dispersion relation is isotropic, indicating that it can propagate in all directions. Its restoring force is magnetic pressure. The Alfvén wave also propagates at velocity v_A but along field lines. It corresponds, as usual, to a noncompressional twisting of the field lines. Finally, the slow wave becomes a wave that propagates along field lines at speed c_s. In this case, the magnetic field (which dominates the pressure) remains fixed, while the gas tied to this fixed field is free only to slide up and down the field lines (1D motion), and thus propagate acoustic waves that are constrained by the magnetic field.

The general behavior of magnetosonic waves when $v_A/c_s \sim 1$ is left as an exercise. It appears that most of the diffuse ISM phases are in this regime (with v_A/c_s slightly greater than unity).

We will need to use Alfvén waves to understand (among other issues) the behavior of turbulent velocity fields in the ISM.

C. PLANE-PARALLEL DISK

Our second example of an MHD solution is a plane-parallel disk, which one may take as a first approximation (dramatically oversimplified!) to the disk of our Galaxy. We suppose there is a gravitational potential $\Phi(z)$ (generated by the stars and gas in the Galactic disk), and that the gas in this disk is stationary with a magnetic field $\mathbf{B}(z)$ and density $\rho(z)$. The mass conservation equation is then trivial.

The momentum conservation equation for a stationary fluid in a gravitational field says that:

$$0 = -\rho\nabla\Phi - \frac{1}{8\pi}\nabla(B^2) + \frac{1}{4\pi}\mathbf{B}\cdot\nabla\mathbf{B} - \nabla P,$$

or

$$0 = \frac{1}{4\pi}B_z\frac{\partial B_x}{\partial z} = \frac{1}{4\pi}B_z\frac{\partial B_y}{\partial z},$$
$$0 = -\rho\frac{\partial\Phi}{\partial z} - \frac{1}{8\pi}\frac{\partial}{\partial z}(B^2) + \frac{1}{4\pi}B_z\frac{\partial B_z}{\partial z} - \frac{\partial P}{\partial z}.$$

Thus we see that there are two possible types of solution: one where $B_z \neq 0$, and B_x and B_y must be constants, and the other where $B_z=0$ and B_x and B_y are allowed to vary. Both have been written down, but since observations of synchrotron polarization show magnetic field structures aligned along the spiral arms of galaxies, the latter is probably closer to physically relevant (although neither is very good; we will improve on this later). In this case, we have:

$$\rho\frac{\partial\Phi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{B^2}{8\pi} + P\right).$$

Thus we see that the gas obeys the usual hydrostatic equilibrium equation, but with a magnetic component to the pressure. Magnetic pressure plays a significant role in the vertical support of the Milky Way's disk.

We have sidestepped the issue here of whether the aforementioned disk is *stable*. In many cases, particularly where magnetic support dominates, it is not. The analysis of MHD solutions for stability is a complicated subject (involving linear perturbation analyses with inhomogeneous backgrounds), and a simple example will appear on the homework.

D. TIME-AVERAGED EQUATIONS FOR DISKS

Of course, the real disk is undergoing turbulent motions and has a complicated magnetic field structure. There would seem to be no way to modify the above beautiful result to take all of this into account. Nevertheless, using the equation of momentum conservation, a useful *statistical* statement can be made about the dynamics of the disk by taking the average of the momentum conservation equation. It should be noted that this procedure does *not* represent a complete description of the disk, but it relies on far fewer approximations than the previous section. In particular, turbulent solutions that result from the long-term evolution of unstable initial conditions can be described.

So now we go ahead and average the momentum conservation equation (assuming the gravitational potential, dominated by the stars, to be fixed):

$$0 = \left\langle \frac{\partial}{\partial t}(\rho v_i) \right\rangle = -\langle \rho \rangle \frac{\partial \Phi}{\partial x_i} - \sum_j \frac{\partial}{\partial x_j} \left[\langle \rho v_i v_j \rangle + \langle P \rangle \delta_{ij} + \langle T_{ij}^{(B)} \rangle \right].$$

This equation has 3 components, but only the z component contains interesting information:

$$\langle \rho \rangle \frac{\partial \Phi}{\partial z} = -\frac{\partial}{\partial z} \left[\langle \rho v_z^2 \rangle + \langle P \rangle + \frac{\langle B_x^2 + B_y^2 - B_z^2 \rangle}{8\pi} \right].$$

Thus at a statistical level the mean density on the left hand side (which represents the gravitational attraction of all matter toward the midplane) must balance the three sources of pressure support that tend to expand the disk: **turbulent pressure** (involving the RMS of the vertical motions); gas pressure (which so far in this class has been **thermal pressure** but could also in general include **cosmic ray pressure**, i.e. the pressure associated with nonthermal particles); and **magnetic pressure**. The last could in principle be negative, but in the real Galactic disk the magnetic field appears to be mainly aligned in the plane of the disk ("horizontal") and hence provides positive support. One of our objectives is to understand how much each source of pressure contributes and develop a quantitative momentum budget for the Galactic disk.