AY 102
HW #6
Due: Wednesday, March 2, 2011

#1. Cooling of cluster gas. [20 points]

Consider a galaxy cluster containing gas at a density of $n \sim 10^{-3}$ cm^{-3}, a mass of 2×10^{14} $M_\odot$, and a temperature of kT = 4 keV. (Real clusters have a range of densities, which complicates the situation, but for the purpose of this problem you can ignore this.)

(a) What are the thermal energy content, free-free luminosity, and free-free cooling timescale of the cluster? In what portion of the electromagnetic spectrum is the free-free radiation emitted?

(b) The preferred ionization stage of iron (which at this temperature is the principal element with bound electrons) at this temperature is Fe XXV. Estimate the lowest excitation energy of Fe XXV (and hence the energy of the principal X-ray line).

(c) Do an order of magnitude estimate of the cross section for electron impact excitation of Fe XXV. What is the expected luminosity in the Fe X-ray line, assuming metal abundances of a few tenths of solar? How does this compare to the free-free luminosity?

#2. Conservation Laws in Incompressible Hydrodynamics and MHD. [20 points]

This series of problems will work you through the proofs of a set of conservation laws. Theorems I and II apply only to hydrodynamics (i.e. assume no magnetic forces). Theorem III applies in MHD.

Steps are provided (but full credit will be assigned for any valid proofs of the theorems).

THEOREM I. *If a fluid is subject to flow only in a gravitational potential, i.e. where* $\mathbf{F}=-\rho\nabla\Psi$, *the medium is incompressible, i.e.* ρ *is a constant, and there is no viscosity, then if C is a closed curve that moves with the fluid, the integral:*

$$\Gamma \equiv \oint_C \mathbf{v}\cdot d\mathbf{x}$$

is conserved.[1] *This integral is called the* **circulation.**

STEPS IN PROOF:

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[1] This is actually true under the more general circumstance that the density ρ is a unique function of the pressure P, but the proof of this more general fact is not required in the problem.

(a) Show that the convective derivative of the velocity may be written as the gradient of some scalar function ϕ:

$$\frac{d\mathbf{v}}{dt} = \nabla\phi.$$

(b) Take the time derivative of Γ. Remember that both **v** and *d***x** have time dependences because the curve *C* is moving. Show that the time derivative can be reduced to the equation:

$$\frac{d\Gamma}{dt} = \oint_C \nabla\left(\phi + \frac{v^2}{2}\right)\cdot d\mathbf{x}.$$

Then use Stokes's theorem to prove that the integral is zero.

THOEREM II. *Consider an incompressible fluid flow in a gravitational potential with no other external forces and zero viscosity, and assume that the velocity field goes to zero sufficiently rapidly at large distances for the integral*

$$H \equiv \int_{\mathbf{R}^3} \mathbf{v}\cdot(\nabla\times\mathbf{v})d^3\mathbf{x}$$

to be well-defined. Then H is conserved. This integral is called the **helicity**.

STEPS IN PROOF:

[*Note*: Since the assumptions of Theorem II are a specialization of those in Theorem I, you may immediately assume that the result from (a) is valid here.]

(c) Show that the integrand:

$$\varpi = \mathbf{v}\cdot(\nabla\times\mathbf{v})$$

satisfies $d\varpi/dt = 0$.

(d) Show that for an incompressible flow, the velocity field has zero divergence. Show that this implies

$$\int_{\mathbf{R}^3} \mathbf{v}\cdot\nabla q \;\; d^3\mathbf{x} = 0$$

for any scalar *q* that vanishes sufficiently rapidly at ∞.

(e) Evaluate *dH*/*dt*, being careful to keep track of the distinction between partial and convective time derivatives.

THEOREM III: *Consider an incompressible, nonviscous, perfectly conducting fluid with no external forces except for magnetic forces, and with a velocity that goes sufficiently rapidly to zero at large distances. Then the* **cross-helicity**

$$H_c \equiv \int_{\mathbf{R}^3} \mathbf{v} \cdot \mathbf{B} \;\; d^3\mathbf{x}$$

is conserved.

(f) Evaluate the convective derivative

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{B}).$$

Show that in the incompressible case, with div **v** = 0, that this can be reduced to $\mathbf{B}\cdot\nabla\alpha$ for some scalar α. Then show that

$$\int_{\mathbf{R}^3} \frac{d}{dt}(\mathbf{v} \cdot \mathbf{B}) \;\; d^3\mathbf{x} = 0.$$

(g) Show that

$$\int_{\mathbf{R}^3} \mathbf{v} \cdot \nabla(\mathbf{v} \cdot \mathbf{B}) \;\; d^3\mathbf{x} = 0.$$

[You may argue that part (d) applies here.]

(h) Evaluate dH_c/dt, being careful to keep track of the distinction between partial and convective time derivatives.