Escape speed

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Abstract

These notes describe the notion of escape speed, which is the topic of a 15-minute lecture given on July 7$^{\text{th}}$ at the University of Birmingham.

From a simple experiment such as throwing a pen in the air, one can see that the fastest an object is launched, the higher it goes up before falling back on the floor. In fact, we will show in this lecture that it is possible to launch an object so fast that it escapes the gravitational attraction of the Earth and never comes back towards it. This leads to the concept of escape speed, which is defined as follows:

The escape speed is the minimum speed needed for an object to escape the gravitational attraction of a planet.

The configuration considered here is illustrated in Fig. 1, which represents an object of mass $m_{\rm o}$ being launched from the surface of a planet of mass $M_{\rm p}$. The trajectory of the object is represented by the red arrow that starts at an initial distance $R_{\rm i}$ from the center of the planet where the initial speed is $v_{\rm i}$. After reaching a final distance $R_{\rm f}$ where the speed is $v_{\rm f}=0$, the object falls back down towards the planet.

To derive a formula for the escape speed, we will use the law of conservation of energy. In this situation, the energy is given by

$$E = m_{\rm o}\frac{v^2}{2} - \frac{GM_{\rm p}m_{\rm o}}{R}, \tag{1}$$

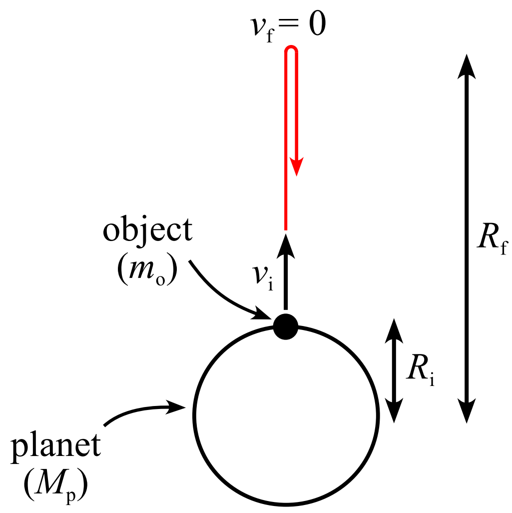

Fig. 1.— Sketch illustrating the configuration considered to derive a formula for the escape speed using the law of energy conservation.

which is the sum of kinetic and potential energy.[1] Here, R and v are the distance and speed of the object at any location on its trajectory.

Conservation of energy implies $E_\mathrm{i} = E_\mathrm{f}$, meaning that the energy is the same at the initial point of the trajectory where the object is launched and at the final point where it has reached the largest distance from the planet. Using equation (1) with the values of both distance and speed defined above, this equality leads to

$$m_\mathrm{o}\frac{v_\mathrm{i}^2}{2} - \frac{GM_\mathrm{p}m_\mathrm{o}}{R_\mathrm{i}} = m_\mathrm{o}\frac{v_\mathrm{f}^2}{2} - \frac{GM_\mathrm{p}m_\mathrm{o}}{R_\mathrm{f}}. \tag{2}$$

The next step is to specify the values of the distance and speed at the initial and final points. The initial speed is equal to the escape speed $v_\mathrm{i} = v_\mathrm{esc}$ that we wish to solve for, while the initial distance is equal to the radius of the planet $R_\mathrm{i} = R_\mathrm{p}$. At the final point, the speed is $v_\mathrm{f} = 0$ while the final radius is set to $R_\mathrm{f} = +\infty$ as the object escapes the gravitational attraction of the planet. Equation (2) then becomes

$$m_\mathrm{o}\frac{v_\mathrm{esc}^2}{2} - \frac{GM_\mathrm{p}m_\mathrm{o}}{R_\mathrm{p}} = 0,$$

which after some algebra yields the following formula for the escape speed:

$$v_\mathrm{esc} = \sqrt{\frac{2GM_\mathrm{p}}{R_\mathrm{p}}}. \tag{3}$$

For the Earth, the planet mass is $M_\mathrm{p} = M_E = 5.97 \times 10^{24}\,\mathrm{kg}$ and its radius $R_\mathrm{p} = R_E = 6.37 \times 10^{6}\,\mathrm{m}$. Using the value of the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, this leads to a value the escape speed for the Earth of $v_\mathrm{esc} = 11.2\,\mathrm{km\,s^{-1}}$, which is larger than that of a bullet by about a factor of 10. It is the speed at which one needs to launch an object to make it escape the Earth's gravity.

Equation (3) implies that decreasing the radius of a planet but keeping its mass fixed results in an increase of the escape speed. If the radius becomes sufficiently small, the escape speed becomes equal to the speed of light c, implying that nothing, not even light, can escape from the gravitational attraction of the planet. Such a compact planet is called a black hole.

Setting the escape speed to $v_\mathrm{esc} = c$ in equation (3) allows us to determine the radius of a black hole, called the Scharzschild radius, which is given by

$$R_\mathrm{S} = \frac{2GM_\mathrm{p}}{c^2}. \tag{4}$$

[1] One can show that this so-called "mechanical" energy is conserved by computing its time derivative, which yields:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m_\mathrm{o}v\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{GM_\mathrm{p}m_\mathrm{o}}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t} = m_\mathrm{o}v\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{GM_\mathrm{p}}{R^2}.\right) = 0$$

Here, we have used $v = \mathrm{d}R/\,\mathrm{d}t$ and the fact that the acceleration of the object times its mass is equal to the gravitational force, that is $m_\mathrm{o}\,\mathrm{d}v/\,\mathrm{d}t = -GM_\mathrm{p}m_\mathrm{o}/R^2$. As expected, $\mathrm{d}E/\,\mathrm{d}t = 0$, which implies that energy is conserved.

For the Earth, using $M_\mathrm{p} = M_E$ and the value of the speed of light $c = 2.99 \times 10^8\,\mathrm{m\,s^{-1}}$ leads to $R_\mathrm{S} = 9\,\mathrm{mm}$, meaning that its mass needs to be compressed inside about the size of sugar cube to become a black hole!

Note that a physically sound derivation of the Schwarschild radius requires solving Einstein's equation of general relativity. This calculation was first carried out by Karl Schwarschild, who gave his name to this radius. The method we used above to obtain equation (4) is therefore only approximate but nevertheless leads to the correct formula. Remarkably, a similar argumentation was used by John Michell to introduce the concept of black hole more than a hundred years before the formulation of the theory of general relativity. Links to these two papers can be found in the bibliography below.

References

- Karl Schwarschild, *"Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie"*, Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin, Klasse fur Mathematik, Physik, und Technik (1916) pp 189 (link to paper)

- John Michell, *"On the Means of Discovering the Distance, Magnitude, &c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should be Found to Take Place in any of Them, and Such Other Data Should be Procured from Observations, as Would be Farther Necessary for That Purpose. By the Rev. John Michell, B. D. F. R. S. In a Letter to Henry Cavendish, Esq. F. R. S. and A. S."*, Philosophical Transactions of the Royal Society of London, The Royal Society, **74**: 35–57, 27 November 1783 (link to paper)