of Stellar Structure, has been elegantly refined and extensively applied by Schwarzschild and Härm and their collaborators at Princeton (see also Bondi and Bondi 1949), and has recently been vigorously pursued for advanced phases of stellar evolution by Hayashi and his colleagues at Kyoto (Hayashi, Hoshi, and Sugimoto 1962). Happily, for our purposes here, as well as for all workers in the field, these techniques have been fully described in Schwarzschild's (1958) lucid monograph, *Structure and Evolution of the Stars*.

The other two approaches have basically consisted of retaining the full constitutive relations in the problem and solving the equations in physical variables on electronic computers. The principal workers, in terms of these two approaches, have been Hoyle (Haselgrove and Hoyle 1956*a*), who has dealt with the differential equations directly and written programs to integrate them by classical techniques such as the Runge-Kutta method, and Henyey (HLL 1955*a*; Henyey, Wilets, Böhm, LeLevier, and Levée [hereafter referred to as HWBLL] 1959; Henyey, Forbes, and Gould [hereafter referred to as HFG] 1964), who has transformed the differential equations explicitly to difference equations and solved them by modern techniques such as relaxation procedures. Since the present writers have some experience with these respective approaches we shall describe them in the next two subsections. Not the least of our motivations has been the feeling that, with fast electronic computers becoming more and more widely available, these potentially powerful tools of astrophysical research (see Wrubel 1960) should be utilized much more extensively than they have been in the past.

3.2. Fitting Method

The construction of evolutionary sequences of stellar models by the method of the present subsection involves separating the problem into a "space part" and a "time part." The first of these is the construction of a single stellar model, that is, the derivation of the march of physical variables between center and surface. The second part is the calculation of the evolutionary time change of some basic characteristics of this model, for example, the change of chemical composition at each point, the work done by gravitational contraction, or the amount of total mass loss. The second part provides the input data for calculating a new model; and the procedure is thence repeated indefinitely to build up an evolutionary sequence of models. We note that it is not at all necessary to choose time as the evolutionary parameter. In certain cases, other variables may be more convenient, such as Schwarzschild's eigenvalue C for radiative envelopes (SES, § 20) or the mass fraction of a growing hydrogen-exhausted core (Hoyle and Schwarzschild 1955). In the techniques described here, however, we shall implicitly restrict the discussion to time t as the evolutionary independent variable, in keeping with a direct physical approach.

In the method of this subsection, the bulk of the work in constructing an evolutionary sequence is in obtaining the space part of the solution, since this

requires solving a non-linear fourth-order boundary-value problem. Linear equations suffice for the time part if we ask only for the time changes in chemical composition at each point in a model, though in cases of fast thermal and dynamical evolution just now coming under attack, the time part will soon become more complex (cf. Hoyle 1959; SES, eq. [12.10]; Sobolev 1960; Sampson 1961; and the following subsection). We shall therefore discuss the space part separately first.

This involves four first-order differential equations of equilibrium, in four dependent variables and one independent variable. It will suffice at the outset to consider any added variables as functions only of the dependent variables and of the chemical composition. The equations for a spherical stellar model have been derived in several expositions (e.g., Wrubel 1958, Schwarzschild 1958, Aller 1954, Chandrasekhar 1951); in conventional notation they are as follows:

$$\frac{dP}{dM_r} = -\frac{G}{4\pi}\frac{M_r}{r^4}, \tag{3.1}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi}\cdot\frac{1}{r^2\rho}, \tag{3.2}$$

$$\frac{dT}{dM_r} = -\frac{3}{64\pi^2 a c}\frac{L_r\kappa}{T^3 r^4} \quad \text{(radiative transfer)}, \tag{3.3a}$$

$$\frac{dT}{dM_r} = \frac{\Gamma-1}{\Gamma}\frac{T}{P}\frac{dP}{dM_r} \quad \text{(convective transport)}, \tag{3.3b}$$

$$\frac{dL_r}{dM_r} = \epsilon. \tag{3.4}$$

These represent, respectively, at each point in a star, hydrostatic equilibrium, conservation of mass, space rate of energy transfer, and conservation of energy. The dependent variables are P, total pressure; r, distance from center; T, temperature; and L_r, energy per unit time emerging from the sphere of radius r. M_r, the mass interior to r, is here taken as the independent variable rather than r, since the dependence of chemical composition on M_r with time is unaltered by expansion or contraction (Haselgrove and Hoyle 1956*a*), both of which occur in virtually all stars during evolution. The constitutive variables, which as noted depend here only on the dependent variables and the chemical composition, are ρ, gas density in mass per unit volume; κ, opacity to radiation in area per unit mass; Γ, adiabatic exponent (Chandrasekhar 1939, pp. 55–59); and ϵ, energy released per unit mass and per unit time. G is the gravitation constant, a is the radiation density constant, and c is the velocity of light (Allen 1963).

Thus, we have four equations in four unknowns, plus the chemical composition. We need four initial conditions or boundary conditions on the dependent variables. For the present discussion of the mathematical structure of the problem it will suffice to use the simplest boundary conditions. These conditions are

not sufficient for models of the sun and cooler stars; reference may be made to SES, § 11, for a more detailed discussion. At the surface, there are natural boundary conditions on pressure and temperature which are generally sufficient for stars hotter than the sun; we have

Use photospheric BC

$$P = 0, \quad T = 0, \quad \text{at } M_r = M, \tag{3.5}$$

where M is the total mass of the star. At the center there are natural boundary conditions on the other two variables:

$$r = 0, \quad L_r = 0, \quad \text{at } M_r = 0. \tag{3.6}$$

With these boundary conditions, and *given* a specified mass, M, and (distribution of) composition, we have sufficient provisions to obtain a solution. The Vogt-Russell theorem (Vogt 1926; Russell, Dugan, and Stewart 1927) asserts further that the solution is unique; however, there is some question whether this has ever been rigorously proven, as pointed out by Odgers (1957). We shall accept Schwarzschild's conclusion of uniqueness, with his caveat with regard to mathematical degeneracy (SES, p. 97), since physically reasonable multiple solutions have not been encountered. (Haselgrove and Hoyle [1956*a*, p. 523] have reported instances of multiple solutions, but with a more general energy-generation expression than $\epsilon = f[\rho, T, \text{composition}]$, which is, of course, basic to the theorem [Chandrasekhar 1939, p. 253].)

To solve the boundary-value problem as stated above, we resort to numerical integration of the differential equations, in the technique of this subsection. Since equations (3.1), (3.2), and (3.3) have singularities at the center, and (3.2) and (3.3) at the surface, it is not possible to proceed all the way from one boundary to the other. (In several cases of interest, however, it is not necessary to do so; see, e.g., Gardiner 1951, Iben and Ehrman 1962.) In general, a "fitting" procedure is required: a pair of trial integrations, one starting from the center and one from the surface, is carried forward until they meet at a common value of the independent variable, the fitting point. If the four dependent variables, respectively from the outward and inward integrations, are continuous across the fitting point, then the boundary-value problem is solved and the model completed. If the dependent variables do not match at the fitting point, as is quite generally the case with the first trial pair, then subsequent trial integrations are necessary. With the boundary conditions as stated, and a given mass and (distribution of) composition, two parameters characterize each integration: P_c and T_c, the central pressure and temperature, characterize the outward trial; and R and L, the radius and luminosity, characterize the inward trial. The procedure may be visualized as a search in a four-dimensional space for a point, with coordinates P_c, T_c, R, and L, which will characterize a pair of integrations having continuity in the dependent variables at the fitting point.

It will be apparent that, in general, the search for a solution will require a large number of trial integrations. The recent successes in the theory of stellar

evolution are due, in large part, to recognition of special circumstances where the number of trial integrations can be ingeniously reduced, as in the classical Cowling (1935; see Schwarzschild 1946; Wrubel 1958, § 40) model, and as illustrated throughout SES. However, we are interested here in a general procedure which will ideally be appropriate for any reasonable stellar model, and which will, moreover, be suitable for automatic computers.

A first attempt at such a procedure has been introduced by Haselgrove and Hoyle (1956*a*). It depends essentially on the fact that there are four dependent variables to be made continuous at the fitting point, and four parameters for each pair of trial integrations. If the rates of change of the dependent variables at the fitting point with respect to each of the four boundary parameters can be found, then in principle it is possible to find a set of boundary parameters which will give continuous dependent variables across the fitting point, as will now be illustrated. This procedure works, with some fiddling.

Denote the four dependent variables by $Y_j (j = 1, \ldots, 4)$. Let the values of these at the fitting point, $M_{r,f}$, from the *outward* trial integration from the center be indicated with superscript o. Let the values at the fitting point from the *inward* trial integration from the surface be indicated with superscript i. For a particular pair of trial integrations, the dependent variables will not be continuous at the fitting point, i.e.,

$$\Delta Y_j = Y_j^i - Y_j^o \neq 0 \qquad (j = 1, \ldots, 4).$$

Denote the four trial parameters at the boundaries by $E_k (k = 1, \ldots, 4)$. $E_1 = P_c$ and $E_3 = T_c$ pertain to the outward trial integration; $E_2 = R$ and $E_4 = L$, to the inward trial integration. Small changes, δE_k, in each one of these will, after a trial integration, produce variations, δY_j, in the dependent variables at the fitting point. Comparison of, say, two trial outward integrations, one with E_1, E_3, and the other with E_1, $E_3 + \delta E_3$, will give the rates of change of the Y_j^o with respect to E_3. A total of six integrations to the fitting point, three outward and three inward, suffices to give the rates of change of the Y_j with respect to the E_k:

$$\frac{\delta Y_j^o}{\delta E_1}, \frac{\delta Y_j^i}{\delta E_2}, \frac{\delta Y_j^o}{\delta E_3}, \frac{\delta Y_j^i}{\delta E_4} \qquad (j = 1, \ldots, 4). \tag{3.7}$$

Figure 3 shows schematically the marches of a particular dependent variable, Y_j, in an outward trial integration, characterized by P_c, T_c, and in an inward trial integration, characterized by R, L. The discrepancy at the fitting point $M_{r,f}$ is indicated by ΔY_j, where $\Delta Y_j > 0$ in the figure. The partial discrepancy in Y_j between the value given by the inward trial and that of the (unknown) solution may be written as $-\Delta^i Y_j$; and similarly for the outward trial we have $+\Delta^o Y_j$ (the minus sign is appropriate to the case illustrated in Fig. 3). The sum of these is the total discrepancy:

$$\Delta Y_j = -\Delta^i Y_j + \Delta^o Y_j. \tag{3.8}$$

Since the inward trial depends on E_2 and E_4, then $Y_j^i = Y_j^i(E_2, E_4)$; and a small change in Y_j^i is related to small changes in E_2 and E_4 by (to first order)

$$\Delta Y_j^i = \frac{\partial Y_j^i}{\partial E_2}\,\Delta E_2 + \frac{\partial Y_j^i}{\partial E_4}\,\Delta E_4 . \tag{3.9}$$

If we identify the left-hand side with the partial discrepancy $\Delta^i Y_j^i$, the quantities ΔE_2 and ΔE_4 are then the changes to be made in the original trial values E_2

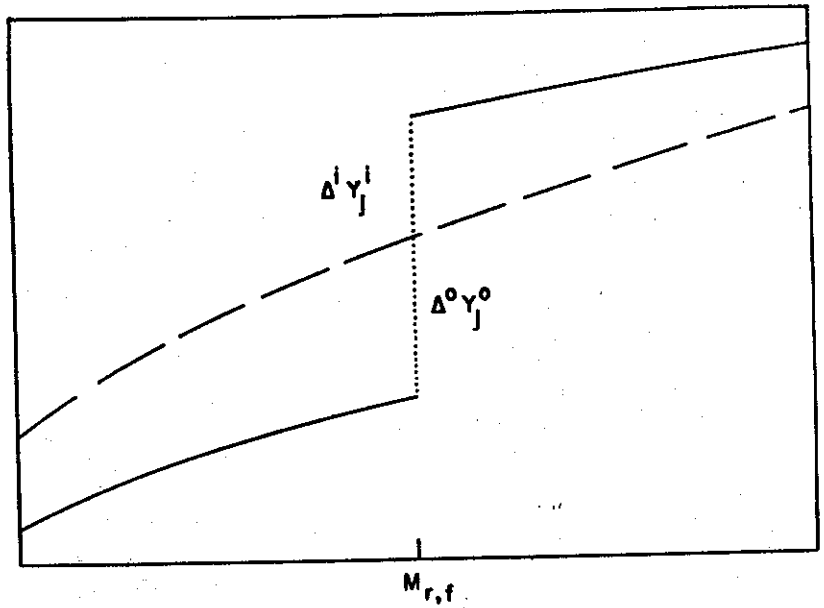

FIG. 3.—Schematic march of Y_j near the fitting point, $M_{r,f}$. Top curve: on an inward trial integration from the surface. Bottom curve: on an outward trial integration from the center. Dashed curve: desirable continuous march of Y_j. Dotted line: total fitting discrepancy, $\Delta Y_j = \Delta^o Y_j^o - \Delta^i Y_j^i$.

and E_4 to produce an inward solution integration, i.e., for Y_j a march along the dashed curve in Figure 3—if the first-order representation of equation (3.9) is sufficient. The partial derivatives are obtained from the appropriate quantities in equation (3.7) above. Thus, equation (3.9) becomes

$$\Delta^i Y_j^i = \frac{\delta Y_j^i}{\delta E_2}\,\Delta E_2 + \frac{\delta Y_j^i}{\delta E_4}\,\Delta E_4 . \tag{3.10}$$

Similarly for the outward integration, we have

$$\Delta^o Y_j^o = \frac{\delta Y_j^o}{\delta E_1}\,\Delta E_1 + \frac{\delta Y_j^o}{\delta E_3}\,\Delta E_3 . \tag{3.11}$$

Hence, from equation (3.8), the discrepancies at the fitting point in each of the dependent variables are given by

$$\Delta Y_j = \frac{\delta Y_j^o}{\delta E_1}\Delta E_1 - \frac{\delta Y_j^i}{\delta E_2}\Delta E_2 + \frac{\delta Y_j^o}{\delta E_3}\Delta E_3 - \frac{\delta Y_j^i}{\delta E_4}\Delta E_4 \quad (j = 1, \ldots, 4), \tag{3.12}$$

$$\Delta Y_j = \sum_{k=1}^{4} (-1)^{k+1}\,\frac{\delta Y_j^{o,i}}{\delta E_k}\,\Delta E_k \qquad (j = 1, \ldots, 4). \tag{3.13}$$

In these four equations the four ΔY_j's are the discrepancies at the fitting point between an original pair of trial integrations. The sixteen derivatives $\delta Y_j^{o,i}/\delta E_k$ may be calculated by varying each of the original trial values by δE_k and, after performing the respective integrations, obtaining the variations δY_j^o or δY_j^i with respect to the values from the original pair. The four equations may then be solved simultaneously for the unknowns, $\Delta E_1, \ldots, \Delta E_4$. Adding these to the original trial values E_k gives new values, E_k', which can be used for a presumably improved pair of integrations.

This method of solving the space part of our over-all problem clearly can be programmed for an automatic computer; an illustrative flow diagram is shown in Figure 4. It will be evident, however, that because of the non-linear character of the differential equations, the convergence implied by the linear approximation of equation (3.9) may even not exist if the first trial pair is very far off in the values of $E_k (k = 1, \ldots, 4)$. Although the procedure has been used successfully in a few cases (Haselgrove and Hoyle 1956*b*, 1958, 1959; Blackler 1958; Hoyle 1959, 1960; Sears 1959, 1960; cf. Schwarzschild and Selberg 1962), no proper analysis of its range of effectiveness has been undertaken. It would be desirable, for example, to investigate various values of the fitting point to see if some one value were peculiarly appropriate for convergence. Such analyses would of necessity require extensive numerical integrations. Experience indicates that computers with add times of the order of a millisecond require from half an hour to an hour for a single pair of integrations; this would probably be too long for an adequate analysis, and a computer with a ten-microsecond add time would be more suitable.

The remaining part of the over-all problem is the time part. Having constructed a single model for a given epoch, we wish to determine the change in the stellar structure over a time interval Δt. The simplest approach, which has proved generally useful in the stages of stellar evolution investigated in recent years, is suggested by the Vogt-Russell theorem: given the mass and the distribution of composition, it is possible to construct a stellar model. Accordingly, if we assume the mass to be constant over the time interval Δt, we can obtain the new distribution of composition at the end of this time from the burning rate ϵ in the previous model. In linear approximation, the new mass fraction of hydrogen at the point M_r in a hydrogen-burning star is given by

$$X_2(M_r) = X_1(M_r) - \left(\frac{\epsilon_1 M_r}{E}\right)\Delta t , \tag{3.14}$$

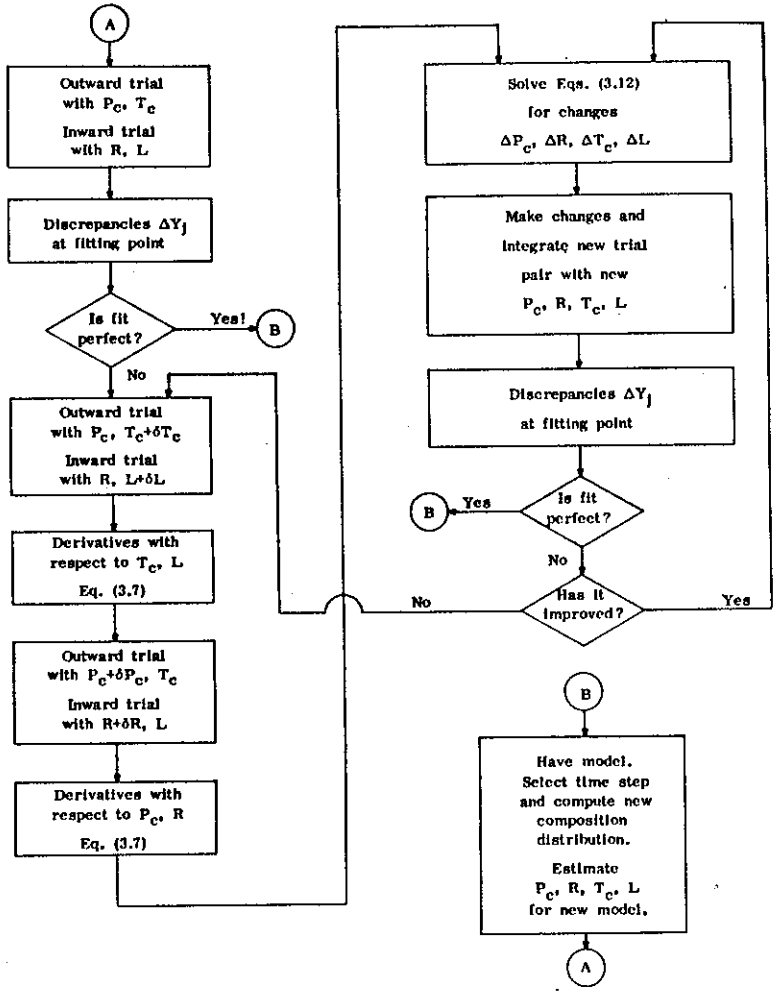

FIG. 4.—Flow diagram for construction of evolutionary sequence of stellar models by fitting trial integrations. Path leading from (A) traces construction of single model. Path leading from (B) indicates how time step is taken between models in an evolutionary sequence.

where subscripts 1 and 2 refer respectively to the previous model (known) and the new model. E is the energy released per unit mass of hydrogen consumed, which is about 6×10^{18} ergs per gram. Equation (3.14), evaluated at each point in model 1, thus gives the new composition at each point in model 2; and one can now return to the space part of the problem and construct model 2 (see Fig. 4). The procedure is repeated indefinitely to build up an evolutionary sequence of stellar models. Equation (3.14) is of course to be supplemented by analogous equations when other nuclear processes than hydrogen-burning have effects on the composition. It may be noted that mixing in a convective region may effectively change the composition homogeneously over a region, in which case an average of equation (3.14) is to be taken over the region (see, e.g., SES, p. 100).

The degree of accuracy of the linear approximation of equation (3.14) depends on the size of the time interval Δt; for too large a value, the implicit assumption of constant burning rate $\epsilon(M_r)$ over Δt will not be satisfactory. In the case of the sun, for example, three steps of $\Delta t = 1.5 \times 10^9$ years, starting from the initial main-sequence state, give hardly different results from a single step of $\Delta t = 4.5 \times 10^9$ years (Sears 1959); but for later stages of evolution it may be expected that the structure will change more rapidly, as witness the step of $\Delta t = 1.6$ *seconds* needed between two of the models undergoing the helium flash, computed by Schwarzschild and Härm (1962). In general, it may be said that a little experience with the method of this subsection in a particular case soon reveals a practical upper limit to the time step, since, for the space part, rather accurate values of the E_k's are needed for the first trials—an automatic computer will eventually "lose" the evolutionary sequence if it is permitted to take too big jumps between models.

As noted in the previous section, it has been realized since the work of Sandage and Schwarzschild (1952) and of HLL (1955*a*) that energy release via gravitational contraction plays a vital role in certain stages of stellar evolution. To see how this is taken into account in the time part of the problem we must generalize the conservation-of-energy equation (3.4) above to include other than nuclear energy production. We start from the first law of thermodynamics:

$$dU = dQ - PdV\,, \tag{3.15}$$

where U = internal energy, Q = heat energy, and PdV = mechanical energy (compression), with P = total pressure and dV = volume change. Let us now specify these quantities per gram. The internal energy of an ideal gas (see, e.g., Limber 1958 for a degenerate gas) is, including radiation energy density,

$$U = \frac{3}{2}\frac{k}{\mu H}T + \frac{aT^4}{\rho} = \frac{3}{2}\frac{P'}{\rho} + \frac{aT^4}{\rho}\,, \tag{3.16}$$

where k = Boltzmann's constant, H = mass of unit molecular weight (μ), a = radiation density constant, and P' = gas pressure. The time rate of change