Ay123 Lecture IX Nuclear Reactions

KWW, Ch. 18

1916: Eddington shows gravitational energy insufficient to power stars

1928: Gamow discovers quantum tunneling allows nuclei to penetrate Coulomb barrier

1939: Bethe discovers pp-chain (1967 Nobel Prize)

1957: Burbidge, Burbidge, Fowler, & Hoyle show most elements synthesized in stars

7.1 Coulomb barrier

Fusion reactions release $\Delta E = \Delta m c^2$, where Δm is the change in mass. Burning $H \rightarrow {}^{56}Fe$ releases $Q/A \sim 8.5$ MeV/nucleon.

But fusing nuclei requires overcoming the Coulomb potential. Is this possible in classical physics?

$$E_{coul} = \frac{Z_1 Z_2 e^2}{r} \sim Z_1 Z_2 \text{ MeV} \quad \text{at } r \sim A^{1/3}(1.4\times10^{-13}\text{ cm})$$

↳ Interaction radius

$$E_{th} = k_B T \sim 1\text{ keV} \quad \text{at } T = 10^7\text{ K}$$

$$E_{coul} \sim 10^3 E_{th}$$

What about particles in tail of Maxwellian distribution?

$$P(v)\,dv = \frac{4\pi m^{3/2}}{(2\pi k_B T)^{3/2}} e^{\frac{-mv^2}{2k_B T}} v^2\,dv$$

$$\Rightarrow P(E)dE \propto e^{-\frac{E}{k_BT}} E\, dE$$

We need particles with $E/k_BT \sim 10^3$, where the probability density is $\sim e^{-10^3} 10^3 \sim 10^{-431}$, so roughly one in 10^{431} protons will have enough energy.

$\# p$ in Sun $\sim 10^{57}$

" in universe $\sim 10^{80}$

Nuclei do not fuse according to classical physics.

Fusion is enabled by quantum tunneling + strong force.

$$F_{nuc} \propto \frac{S}{r^2} e^{-r/r_0}$$

$$r_0 \simeq r_{nuc} \sim A^{1/3}(1.4 \times 10^{-13}\text{ cm})$$

Note 10^{-13} cm = 1 fermi, $(10\text{ fermi})^2 = 10^{-24}\text{ cm}^2 = 1$ barn

Approx cross section of ^{238}U

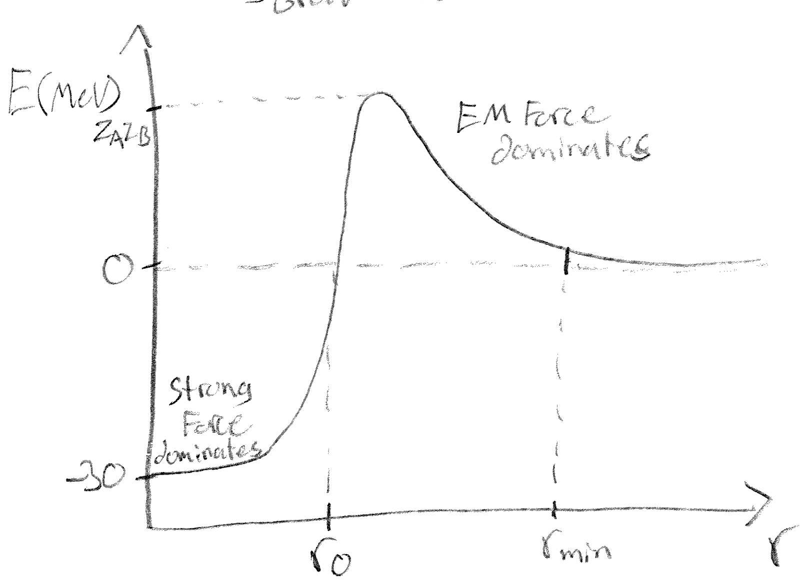
S is strong force coupling.

$$S_{EM} \sim \alpha \sim \frac{1}{137}$$

$$S_{weak} \sim 10^{-21}$$

$$S_{strong} \sim 10$$

$$S_{Grav} \sim 10^{-42}$$



Comparing thermal E to EM E,

$$\frac{Z_AZ_Be^2}{r_{min}} \sim k_BT$$

$$\Rightarrow r_{min} = \frac{Z_AZ_Be^2}{k_BT}$$

$$\simeq 1.7\, Z_AZ_B \times 10^{-10}\text{ cm}\left(\frac{T}{10^7K}\right)^{-1}$$

$$\sim 10^3 r_0$$

Fusion requires $T \sim 10^{10}$ K classically

7.2 Quantum Tunneling

QM allows finite probability for a nucleon to penetrate through the Coulomb barrier.

Probability of tunneling distance r_{min} is

$$\sigma(E) \sim \frac{S}{E} e^{-\frac{r_{min}}{\Delta x}}$$

From Heisenberg uncertainty principle,

$$\Delta x \Delta p \sim \hbar$$

$$\Rightarrow \Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{p} \sim \frac{\hbar}{\sqrt{2mE}} \sim \lambda\!\!\!^{-} \quad \text{de Broglie Wavelength}$$

So

$$\sigma(E) \sim \frac{S}{E} e^{-\frac{\sqrt{2mE}\, Z_A Z_B e^2}{\hbar E}} \sim \frac{S}{E} e^{-\frac{\sqrt{m}\, Z_A Z_B e^2}{\hbar \sqrt{E}}}$$

Reaction rate scales as

$$P \propto \langle \sigma v \rangle \qquad m = \frac{m_A m_B}{m_{AB}} = \text{Reduced Mass}$$

$$\propto \frac{S}{m^{1/2} E^{1/2}} e^{-\frac{m^{1/2} Z_A Z_B e^2}{\sqrt{E}}}$$

Cross sections affected by resonances w/ excited states

Consider reaction

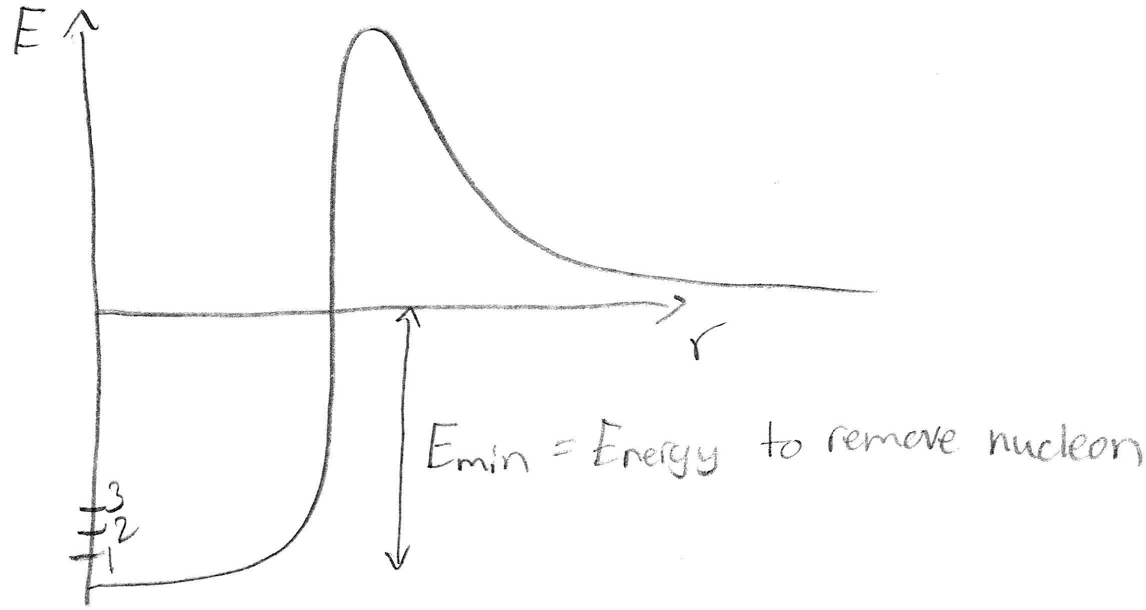
$$\underbrace{X + a \rightarrow Y}_{\text{Nuclei}} + b$$

Initial step can create excited state

$$a + X \rightarrow C^* \qquad (\text{e.g., } p + p \rightarrow {}^2He)$$

C^* can decay, e.g.,

$$C^* \rightarrow X + a, \quad C^* \rightarrow \underbrace{Y_1}_{\text{different atom}} + b_1, \quad C^* \rightarrow C + \gamma$$



If $E < E_{min}$, only γ emission possible, not enough energy to remove nucleon and cause nuclear decay.

Energy width of decay is, from uncertainty principle,

$$\Gamma = \frac{\hbar}{\tau} \leadsto \text{lifetime}$$

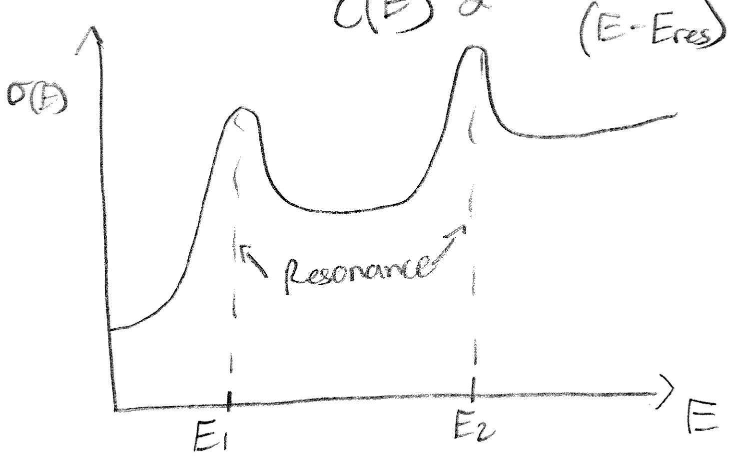
For $E > E_{min}$, particle emission can occur

In general, cross section is

$$\sigma(E) \sim \pi \lambda\!\!\!^{-2}\, \underbrace{P(E)}_{\text{Tunneling}}\, \underbrace{\xi(E)}_{\text{Resonance Factor}}$$

$$P(E) \propto E^{-1/2} e^{\frac{-m^{1/2} Z_A Z_B e^2}{\hbar E^{1/2}}}$$

$$\xi(E) \propto \frac{1}{(E - E_{res}) + (\Gamma/2)^2}$$



Reaction cross sections (including effects of resonances) can be expressed as

$$\sigma(E) = \underbrace{\frac{S}{E}}_{\text{"Astrophysical S"}} e^{-2\pi\eta} \qquad \eta = \frac{m^{1/2} Z_A Z_B e^2}{\hbar E^{1/2}}$$

"Astrophysical S"
Depends on nuclear properties,
usually measured rather than calculated.

For pp chain, at $T = 10^7$ K, $P \sim 10^{-20}$, and proton lifetime is $\sim 10^{10}$ yr.

7.3 Nuclear Cross Sections

$$\sigma_{AB} = \frac{\#\text{ Reactions per time per target A}}{\text{Incident flux of projectiles B}} \quad \text{cm}^2$$

Projectile Flux $= n_B v$

Total Reaction Rate:

$$R_{AB}(v) = n_A n_B \sigma_{AB}(v)\, v \qquad (\text{cm}^{-3}\,\text{s}^{-1})$$

Divide by 2 if A=B to avoid double counting

In most cases, v has Maxwell-Boltzmann distribution. We want to know

$$\langle \sigma v \rangle = \text{Average over MB distribution}$$

Energy generation rate:

$$\epsilon = \frac{Q_{AB} \langle R_{AB} \rangle}{\rho} \quad (\text{erg g}^{-1}\,\text{s}^{-1}) \qquad Q_{AB} = \Delta m c^2$$

= Energy per reaction

7.4 Non-resonant reactions

$\langle \sigma v \rangle$ is integral over MB distribution:

$$f(E) = \frac{2}{\pi^{1/2}} (k_B T)^{-3/2} e^{-E/k_B T} E^{1/2}$$

and tunneling probability

$$\sigma(E) = \frac{S}{E} e^{-2\pi\eta}$$

where $E = \frac{mv^2}{2} \Rightarrow v = \left(\frac{2E}{m}\right)^{1/2}$, $\eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_A Z_B e^2}{\hbar E^{1/2}}$

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E)\, v f(E)\, dE$$

$$= \left(\frac{8}{\pi m}\right)^{1/2} \frac{1}{(k_B T)^{3/2}} S \int_0^\infty e^{-\frac{E}{k_B T}} e^{-\frac{B}{E^{1/2}}} dE$$

$$B = \pi (2m)^{1/2} \frac{Z_A Z_B e^2}{\hbar}$$

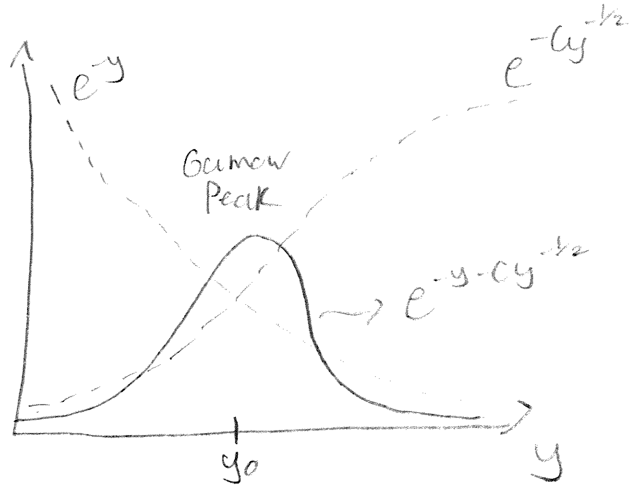
Let $y = E/k_B T$,

$$\langle \sigma v \rangle = \left(\frac{8}{\pi m}\right)^{1/2} \frac{1}{(k_B T)^{1/2}} S \int_0^\infty e^{-y - cy^{-1/2}} dy$$

In HW, solve for value of y_0, corresponding to energy

$$E_{max} \sim 1\,\text{keV} \left(Z_A Z_B T_6\right)^{2/3}$$

$$T_6 = \frac{T}{10^6 K}$$



For pp-chain, $Z_A = Z_B = 1$, $T_6 \sim 10$,

$\Rightarrow E_{max} \sim 4.5$ keV

In sun, $E_{th} = \frac{1}{2} k_B T \sim 0.6$ keV $\Rightarrow E_{max}/E_{th} \sim 7$ Reactions occur in tail of MB dist

At Gamow peak,

$$W \equiv Z_A^2 Z_B^2 A$$

$$\tau \simeq 20\, W^{1/3} T_7^{-1/3}$$

reaction rate scales with temperature as

$$\nu = \frac{\partial \ln \langle \sigma v \rangle}{\partial \ln T} = \frac{\tau - 2}{3}$$

For pp-chain, $\nu \sim 4$