Ay 123 Lecture III. Equations of Stellar Structure

Because τ_{dyn} is short compared to evolutionary timescales, we look for equilibrium time-independent equations.

3.1 Hydrostatics

Hydrostatic Equilibrium: $$\boxed{\frac{dP}{dr} = -\frac{Gm\rho}{r^2}} \quad 3.1$$

We can write this as

$$\frac{dP}{dr} = -\rho g$$

$$\frac{dP}{P} = \frac{-\rho g}{P}\,dr$$

$$= \frac{-dr}{H} \quad \text{where } H = \frac{P}{\rho g} \text{ is the scale height,}$$

i.e., the length scale over which the pressure changes. For an ideal gas, $H = \frac{k_B T}{\mu m_p g}$

In stellar atmospheres, T and g are roughly constant, and H is the scale height on which P and ρ exponentially decline.

We also have the mass equation $$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho} \quad 3.2$$

We need equation of state to relate P and ρ to solve for stellar structure.

3.2 Polytropes

Polytropic equation of state hees

$$\boxed{P = K\rho^{\gamma}} \quad 3.3$$

$K \to$ constant

$$\gamma = \frac{C_P}{C_V} \qquad C_P = \left.\frac{\partial U}{\partial T}\right|_P \qquad C_V = \left.\frac{\partial U}{\partial T}\right|_V$$

$= \frac{5}{3}$ for ideal gas

1st Law of Thermodynamics

$$\delta U = \delta Q - P\delta V$$

★ In real stars, although $\gamma = 5/3 =$ const is often good approximation, K is not generally constant, i.e. is a function of radius, K also varies for different stars

From Eq. 3.1,

$$m = \frac{-r^2}{G\rho}\frac{dP}{dr}$$

$$\Rightarrow \frac{dm}{dr} = \frac{-d}{dr}\left[\frac{r^2}{G\rho}\frac{dP}{dr}\right] = 4\pi r^2\rho$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}K\frac{d}{dr}\rho^{\gamma}\right] = -4\pi G\rho$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left[\gamma K r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

Second order ODE for ρ. Boundary condition:

$$\rho = \rho_c \text{ at } r = 0$$

$$\rho = 0 \text{ at } r = R$$

Define

$$\rho = \lambda\theta^n$$

$\lambda \to$ constant w/ units density

$\theta^n \to$ dimensionless variable

with $n = \frac{1}{\gamma - 1}$

$\gamma = 1 + \frac{1}{n}$

Then we have

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{n+1}{n}K r^2 (\lambda\theta^n)^{\frac{1-n}{n}} \frac{d}{dr}(\lambda\theta^n)\right] = -4\pi G\lambda\theta^n$$

$$\Rightarrow \underbrace{\left(\frac{n+1}{4\pi G}K\lambda^{\frac{1-n}{n}}\right)}_{\equiv \alpha^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Note α has units of length

Now define $r = \alpha\xi$ ↳ dimensionless variable

$$\Rightarrow \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$$ Lane-Emden Equation

ξ = Dimensionless radius

θ = Dimensionless density

For given n, solve equation and then convert $\theta(\xi) \to \rho(r)$.
Each solution has two parameters K and λ.

Boundary conditions

$$\rho_c = \lambda\theta_c^n \quad \text{at } r=\xi=0$$

Convenient to choose $\lambda = \rho_c$ so that BC becomes

$$\theta_c = 1 \quad \text{at } \xi = 0$$

Note also that $\frac{dP}{dr} = -\rho g \propto \frac{d\theta}{d\xi}$. At center of star, $g \to 0$, so we can also choose

$$\frac{d\theta}{d\xi} = 0 \quad \text{at } \xi = 0$$

Lane-Emden equation can be numerically integrated for any n. There are analytic solutions for

$$n=0: \quad \theta = 1 - \frac{\xi^2}{6}$$

$$n=1: \quad \theta = \frac{\sin\xi}{\xi}$$

$$n=5: \quad \theta = \left(1 + \xi^2/3\right)^{-1/2}$$

The outer BC is $\theta = 0$ at $\xi = \xi_1$, $R = \alpha \xi_1$

$$n=0 \rightarrow \xi_1 = \sqrt{6}$$
$$n=1 \rightarrow \xi_1 = \pi$$
$$n=5 \rightarrow \xi_1 \rightarrow \infty$$

The star's radius is

$$R = \alpha \xi_1 = \left[\frac{n+1}{4\pi G} K \rho_c^{\frac{1-n}{n}}\right]^{1/2} \xi_1$$

Note R is independent of ρ_c for $n=1$, i.e. R is a fixed radius set by K

and mass is

$$M = 4\pi \alpha^3 \rho_c \left(-\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

$$= 4\pi \left(\frac{n+1}{4\pi G} K\right)^{3/2} \rho_c^{\frac{3-n}{2n}} \left(-\xi^2 \frac{d\theta}{d\xi}\right)_{\xi_1}$$

Note M is independent of ρ_c for $n=3$, i.e., M is a fixed mass set by K.

Specific cases

$n=0$: constant density, incompressible meeterial
- low-mass planets/moons made of water/rock

$n=1$: Radius independent of mass
- Gas giants to brown dwarfs, neutron stars?

$n=3/2$: Non-relativistic degeneracy pressure or convecting ideal gas
- white dwarfs, low-mass stars, red giant stars

$n=3$: Mass independent of radius
- Chandrasekhar mass relativistic degeneracy pressure white dwarfs
- radiation-pressure supported star

$n \geq 5$: Unstable

$n \to \infty$: Isothermal ideal gas

If the ideal gas law applies,

$$\rho = \rho_c \Theta^n$$

$$\Rightarrow \rho^\gamma = \rho_c^\gamma \Theta^{n\gamma}$$

$$\Rightarrow P = P_c \Theta^{n\gamma}$$

$$\Rightarrow \rho T = \rho_c T_c \Theta^{n\gamma}$$

$$\Rightarrow T = T_c \Theta^{n(\gamma - 1)}$$

$$\Rightarrow \boxed{T = T_c \Theta}$$

Note for mixture of ideal gas and radiation,

$$P_{gas} = \frac{\rho k_B T}{\mu m_p} \equiv \beta P$$

$$P_{rad} = \frac{1}{3} a T^4 = (1-\beta) P$$

Eliminating T,

$$\beta^4 P^4 \left(\frac{\mu m_p}{\rho k_B}\right)^4 = \frac{3(1-\beta)P}{a}$$

$$\Rightarrow P = \left(\frac{k_B}{\mu m_p}\right)^{4/3} \left[\frac{3(1-\beta)}{a\beta^4}\right]^{1/3} \rho^{4/3}$$

This is the Eddington model for high-mass stars. Only a polytrope if β = constant.

3.3 Energy Transport

In steady state,

$$\frac{dL}{dr} = 4\pi r^2 \epsilon$$

Luminosity

Energy generation per unit volume per unit time

Or

$$\boxed{\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon} \quad 3.4$$

Energy generation per unit time per unit mass

Energy generation by fusion is very sensitive to temperature, and slightly dependent on density

$$\varepsilon = \varepsilon(\rho, T)$$

From thermodynamics, energy flow from high to low T, (core to surface)

Energy can be carried by

- radiation (photons, neutrinos)
- conduction (atomic motion)
- convection (bulk motion)
- hydrodynamic waves (oscillatory motion)

Radiation and conduction are diffusive processes
Convection not as well understood
waves only important in specific circumstances

Radiation Transport

Photon mean-free path is $l = \frac{1}{\rho \kappa}$ ↳ Opacity, units $= \frac{cm^2}{g}$

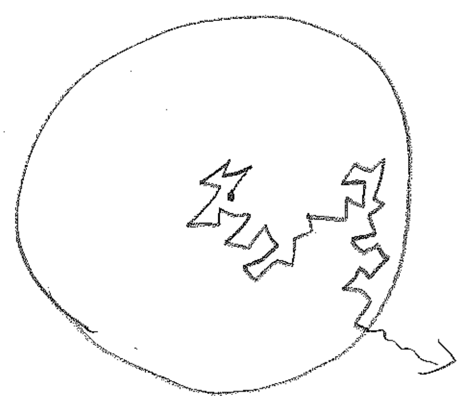
Inside Sun, $\kappa \sim 10$ cm^2/g, $\rho \sim 1$ g/cm^3, $l \sim 0.1cm$

Photons must random walk out from core of Sun in 0.1 cm sized steps.

How long does it take photons to work their way out?

Hint: For random walk of step size l to travel distance d, number of steps expected is

$$N = \left(d/l\right)^2$$



For photons, the duration of each step is

$$t_{step} = \frac{\ell}{c}$$

So the time to escape is

$$t_{esc} = N\, t_{step} = \frac{d^2}{\ell c}$$

For Sun, using $d = 7\times 10^{10}$ cm, $\ell = 0.1$ cm, we have

$$t_{esc} \sim 5\times 10^4 \text{ yr}$$

Alternatively, the diffusivity of photons is $D \sim c\ell$, and diffusion time is $t_{diff} = d^2/D = d^2/\ell c$.

Note anisotropy of radiation field is tiny

$$u \propto T^4$$

$$\Rightarrow \frac{du}{u} = \frac{4\,dT}{T} = \frac{4}{T}\frac{dT}{dr}\,dr$$

$$\sim \frac{4}{T_c}\frac{T_c - T_s}{R_\odot}\,\ell \sim \frac{4}{R_\odot}\,\ell \sim 10^{-12} \quad \text{Tiny!}$$

Since ℓ is much smaller than R, H, we can approximate radiative transport by a diffusion equation

$$\underbrace{\vec{F}_{rad}}_{\text{Radiative Flux}} = -\underbrace{D}_{\text{Diffusivity}}\,\vec{\nabla}\underbrace{U_{rad}}_{\text{Radiation energy density}}$$

where $D = \frac{1}{3}v\ell = \frac{c}{3\kappa\rho}$ for photons and $U_{rad} = aT^4$. In spherical stars, $\vec{\nabla}$ has only radial dependence

So $$\frac{\partial U_{rad}}{\partial r} = 4aT^3\frac{\partial T}{\partial r}$$

$$\Rightarrow F_{rad} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r}$$

$$= -K_{rad}\nabla T$$

↳ Coefficient of radiative conduction

Using $L_{rad} = 4\pi r^2 F_{rad}$, we have

$$\boxed{\frac{\partial T}{\partial r} = \frac{-3}{16\pi ac}\frac{\kappa\rho L_{rad}}{r^2T^3}}$$

In radiative zones in stellar interiors, $L_{rad} \sim L$. In convection zones, however, $L_{rad} \ll L$ in most cases.