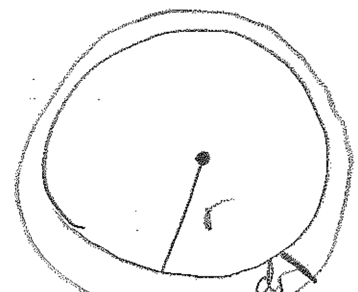
Ay 123 Lecture II Equilibrium & Timescales

2.1 Hydrostatic Equilibrium

A star in equilibrium experiences no internal accelerations.
So Newton's Law is

$$\vec{F} = m\vec{a} = 0$$

Consider a spherical shell of thickness dr



It experiences two forces:

Gravity: $F_g = \frac{-Gm\,dm}{r^2}$

And mass of shell is

$$dm = 4\pi r^2 \rho\, dr$$

$$\Rightarrow F_g = -4\pi G m \rho\, dr$$

Shell experiences force due to pressure gradient across shell

$$F_p = -dP\,A = -4\pi r^2 dP$$

So equilibrium requires

$$F_g + F_p = 0$$

$$\Rightarrow -4\pi r^2 dP - 4\pi G m \rho\, dr = 0$$

$$\Rightarrow \frac{dP}{dr} = \frac{-Gm}{r^2}\rho \Rightarrow \boxed{\frac{dP}{dr} = -\rho g}$$ Hydrostatic Equilibrium

Where $g = \frac{Gm}{r^2}$ is the gravitational acceleration.
Note this can be written

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

2.2 Energy

The gravitational binding energy of a spherical shell is

$$d\Omega = -\int_r^\infty \frac{Gm(r)\,dm(r)}{r'^2}\,dr'$$

$$= -\frac{Gm(r)\,dm}{r}$$

So the total binding energy is

$$\Omega = \int d\Omega = -\int_0^M \frac{Gm(r)}{r}\,dm$$

$$= -\alpha \frac{GM^2}{R}$$

$\hookrightarrow$ constant that depends on structure of star

Note that the gravitational binding energy is negative by definition.
There is also internal energy of the star

E = internal energy per unit mass

$U = \int E\,dM$ = total internal energy

So the total energy of the star is

$$W = U + \Omega$$

Hydrostatic equilibrium can also be derived by setting $dW = 0$, i.e., requiring system to be in a minimum energy configuration, see HKT pp. 5-7.

2.3 Virial Theorem

Hydrostatic equilibrium is

$$\frac{dP}{dm} = \frac{-Gm}{4\pi r^4}$$

$$\Rightarrow dP\,V = \frac{-Gm\,dm}{4\pi r^4} V$$

where $V = \frac{4\pi}{3} r^3$ = volume, Integrate

$$\int V dP = \int \frac{-Gm\,dm}{3r} = \frac{1}{3}\Omega$$

Note

$$\int V dP = \int d(VP) - \int P dV$$

$$= VP\Big|_{r=0}^{r=R} = -\int P dV$$

$$= -\int P dV$$

So

$$\boxed{-3\int P dV = \Omega}$$ Virial Theorem

Forms of pressure

i. Ideal gas pressure

Many stars well-approximated by ideal gas

Ideal gas law:

$$P = n k_B T \qquad n = \frac{\rho}{\mu m_p} = \text{number density of particles}$$

k_B → Boltzmann constant

μ → μ = mean molecular weight per proton mass

$$\boxed{P = \frac{\rho k_B T}{\mu m_p}}$$

Can also be written $P = \rho(c_p - c_v)T = \rho(\gamma - 1) c_v T = \rho(\gamma-1)E$

with c_p, c_v specific heats and $\gamma = c_p/c_v$.

For monoatomic ideal gas, $c_v = \frac{3}{2}\frac{k_B}{\mu m_p}$, $c_p = \frac{5}{2}\frac{k_B}{\mu m_p}$, $\boxed{\gamma = \frac{5}{3}}$

And the kinetic energy per particle is $\frac{3}{2} k_B T$

So the internal energy per unit mass is

$$E = \frac{3 k_B T}{2 \mu m_p} = \frac{3}{2}\frac{P}{\rho}$$

Note that for ideal gas, virial theorem is

$$-3\int P \, dV = \Omega$$

$$= -3\int \frac{2}{3}\rho E \, dV = \Omega$$

$$= -2\int \varepsilon \, dV = \Omega$$

where ε is internal energy per unit volume

$$\Rightarrow \boxed{-2U = \Omega}$$

ii. Radiation pressure

From Planck function, number density of photons in frequency interval $d\nu$ is

$$n(\nu)\, d\nu = \frac{8\pi \nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} d\nu$$

And the pressure exerted is

$$P = \frac{1}{3}\int \underbrace{v}_{\text{velocity}} \underbrace{p}_{\text{momentum}} n(p)\, dp$$

For photons, $c = v$, and $p = \frac{h\nu}{c} \Rightarrow dp = \frac{h}{c} d\nu$

$$\Rightarrow P = \frac{1}{3} h \int_0^\infty \nu\, n(\nu)\, d\nu$$

Evaluates to $\boxed{P_{rad} = \frac{1}{3} a T^4}$ $\quad a = \frac{4\sigma_B}{c} = \frac{8\pi^5 k_B^4}{15c^3h^3}$

For radiation the energy density is $\mathcal{E} = aT^4$ and $\boxed{\gamma = \frac{4}{3}}$

And the virial theorem yields

$$-3\int P\, dV = \Omega$$

$$\Rightarrow -\int \mathcal{E}\, dV = \Omega$$

$$\Rightarrow \boxed{-U = \Omega}$$

2.4 Central temperature and pressure

Let's assume ideal gas EOS, then

$$\Omega = -\alpha \frac{GM^2}{R}$$

$$U = \frac{-\alpha}{2}\frac{GM^2}{R} = \int_0^M \frac{3}{2}\frac{k_B T}{\mu m_p}\, dm = \frac{3}{2}\frac{k_B \bar{T}}{\mu m_p} M$$

mass-averaged temperature

and $\bar{\rho} = \frac{3M}{4\pi R^3}$

So we see that

$$\bar{T} \propto \frac{M}{R} \propto M^{2/3} \bar{\rho}^{-1/3}$$

So higher mass implies hotter temperatures. We can expect internal temperatures

$$\bar{T} \sim \frac{GM\mu m_p}{k_B R} \sim 5\times10^6 \text{ K}$$

Accurate model has central temperature $T_c \sim 1.5\times10^7$ K

Aside: μ is mean molecular weight per particle. For ionized hydrogen,

$$\mu = \frac{1}{m_p}\left(\frac{m_p + m_e}{2}\right) \simeq \frac{1}{2}$$

For ionized helium,

$$\mu = \frac{1}{m_p}\frac{m_{He} + 2m_e}{3} \simeq \frac{4}{3}$$

For ionized metals $\mu \simeq 2$

2.5. Timescales

i. Dynamical (free-fall) timescale

In the absence of pressure,

$$\frac{\partial^2 r}{\partial t^2} = -g = \frac{-GM}{r^2}$$

$$\Rightarrow t_{dyn} \sim \frac{R^3}{GM} \Rightarrow t_{dyn} \sim \sqrt{\frac{R^3}{GM}} \sim 30 \text{ min for Sun}$$

Alternatively, if we removed gravity, we would have

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{dP}{dr}$$

$$\Rightarrow t_{exp}^2 \sim \frac{R^2\rho}{P} \sim \frac{R^2}{c_s^2} \qquad \text{since } c_s^2 = \frac{\gamma P}{\rho}$$

sound speed

$$\Rightarrow \boxed{t_{exp} \sim R/c_s}$$

soundspeed crossing timescale

Since stars are in hydrostatic equilibrium, the expansion timescale is comparable to the dynamical time scale.

ii. Kelvin-Helmholtz timescale

$$\tau_{KH} = \frac{\Omega}{L} \sim \frac{GM^2}{RL} \sim 3\times10^7 \text{ yr for Sun}$$

↳ Luminosity

Timescale for contraction if luminosity is generated by gravitational contraction.

Note, from virial theorem, $\Omega \sim U$, so KH timescale is also the timescale to radiate away a star's thermal energy, ie, it is a <u>thermal timescale</u>

τ_{KH} is comparable to the pre-main sequence lifetime of a star. Because $\tau_{KH} \gg \tau_{dyn}$, stellar oscillations are nearly adiabatic

iii. Nuclear timescale

$$\tau_{nuc} = \frac{E_{nuc}}{L}$$

Energy content available:

$4\,H \rightarrow He$ releases $Q = 6.3 \times 10^{18}$ erg/g

$\Rightarrow E_{nuc} \sim MQ$

$\Rightarrow \tau_{nuc} \sim \frac{MQ}{L} \sim 10^{11}$ yr for Sun

actual lifetime of sun is $\sim 10^{10}$ yr because only hydrogen in core will be burned.