Ay123 Set 5 solutions

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1. **Supernova Shock Revival from Neutrino Heating**

(a) **Estimate the energy that is required to photodissociate 0.8 $M_\odot$ of Fe into alpha particles and neutrons. Compare this energy to the bounce shock energy and comment on the fate of the shock.**

First find what particles ^{56}Fe dissociates into: $^{56}\text{Fe} = 26\text{p} + 30\text{n} = 13\alpha + 4\text{n}$

Then find the energy needed to dissociate a single ^{56}Fe nucleus:

$$\begin{aligned} Q &= (13m_\alpha) + 4m_n - m(^{56}\text{Fe})c^2 \\ &= (56 - 55.85)m_p c^2 \\ &= 2.25 \times 10^{-4} \text{ erg/nucleus} \end{aligned}$$

Now calculate the total energy needed to photodissociate 0.8 $M_\odot$ of ^{56}Fe:

$$\begin{aligned} E_{\text{phot}} &= Q(0.8\ M_\odot) \\ &= (2.25 \times 10^{-4} \text{ erg}/\cancel{\text{nucleus}})(0.8\ \cancel{M_\odot})(2 \times 10^{33}\ \cancel{\text{g}}/\cancel{M_\odot})(55.85 \times 1.67 \times 10^{-24}\ \cancel{\text{g}}/\cancel{\text{nucleus}})^{-1} \end{aligned}$$

This yields $\boxed{E_{\text{phot}} = 3.9 \times 10^{51} \text{ erg}}$. The photodissociation energy is larger than the energy of the bounce $E_{\text{bounce}} = 10^{51}$ erg, so the shock will not survive with its initial energy.

(b) **In the proto-neutron star (with an initial radius 2×10^6 cm), the mean free path of neutrinos is $l_\nu = 30$ cm. Estimate the diffusion time for neutrinos to escape from the proto-neutron star and hence estimate the neutrino luminosity during the initial neutron-star cooling phase.**

The diffusion time is given by $t_{\text{diff}} = \frac{R^2}{lc}$, where $R = 2 \times 10^6$ cm is the radius and $l = 30$ cm is the mean free path. Plugging in numbers yields $\boxed{t_{\text{diff}} = 4.44 \text{ s}}$.

The neutrino luminosity L_ν is generated by the neutrinos radiating the proto-neutron star's gravitational binding energy $E_{\text{bind}} \sim \frac{GM_{\text{core}}^2}{R}$. Then $L_\nu \sim E_{\text{bind}}/t_{\text{diff}}$. Plugging in numbers yields $\boxed{L_\nu = 1.5 \times 10^{52} \text{ erg}}$.

(c) **Assuming that 10% of the neutrino luminosity is absorbed by the infalling outer core, estimate how long it takes to absorb enough neutrino energy to reverse the infall of the 0.8 $M_\odot$ outer core and drive a successful supernova explosion with a typical explosion energy of 10^{51} erg. Assume the outer core has initial energy per unit mass $\epsilon = -GM_{Fe}/R_{Fe}$. Compare this time to the dynamical (free-fall) timescale of the proto-neutron star.**

Assuming 10% of the neutrino luminosity is absorbed by the core, the total energy absorbed is $0.1L_\nu t$:

$$0.1L_\nu t = E_{\text{infall}} + E_{\text{SN}}. \tag{1}$$

Here, $E_{\rm SN} = 10^{51}$ erg is the energy of the supernova and $E_{\rm infall}$ is the infall energy of the outer core. The total infall energy is given by

$$\begin{aligned} E_{\rm infall} &= \epsilon M_{\rm outer\ core} \\ &= \frac{GM_{\rm Fe}}{R_{\rm Fe}}(0.8\ M_\odot) \\ &= 1.07 \times 10^{51}\ \text{erg} \end{aligned}$$

Solving equation (1) for the time, we find

$$\begin{aligned} t &= \frac{E_{\rm infall} + E_{\rm SN}}{0.1 L_\nu} \\ &= \frac{(1.07+1)\times 10^{51}\ \text{erg}}{0.1(1.5\times 10^{52}\ \text{erg/s})} \end{aligned}$$

which yields $\boxed{t = 1.4\ \text{s}}$.

Compare this to the dynamical (free-fall) time of $\boxed{t_{\rm dyn} = \sqrt{\frac{R^3}{GM}} = 2.9\times 10^{-4}\ \text{s}}$. It takes many dynamical timescales to drive the explosion!

2. **Protostar**

   (a) **Find the average density and central temperature (as a function of mass) of an accreting protostar whose initial radius is given by the expression**

$$\frac{R}{R_\odot} = \frac{43.2}{1-0.2X}\frac{M}{M_\odot}$$

   if its structure is approximated by a $n = 1.5$ polytrope with hydrogen mass fraction $X = 0.7$ and helium fraction $Y = 0.3$.

   The average density is given by $\bar{\rho} = \frac{3}{4\pi}\frac{M}{R^3}$. We know that the radius is given by $R = \frac{43.2}{1-0.2X}\frac{M}{M_\odot}R_\odot$. Substituting $X = 0.7$ and the solar values, we find that $R = (1.76\times 10^{-21}\ \text{cm/g})M$. The average density is then given by

$$\bar{\rho} = \frac{3}{4\pi}\frac{M}{(1.76\times 10^{-21}\ \text{cm/g})^3 M^3}$$

$$\boxed{\bar{\rho} = (1.1\times 10^{-5}\ \text{g/cm}^3)\left(\frac{M}{M_\odot}\right)^{-2}}$$

   The central temperature is given by $T_c = C\frac{\mu M}{R}$ where $C = 4.347\times 10^{-16}$ (cf. Set 4 or HKT Eq. 7.41). One of the easiest ways to compute mean molecular weight is using $\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$, which yields $\mu = 0.62$. Plugging in numbers (fortunately M appears in both the numerator and denominator and cancels out), we find $\boxed{T_c = 1.52\times 10^5\ \text{K}}$.

   (b) **Suppose the protostar maintains a polytropic structure until its collapse is halted when the central temperature reaches $T_{\bf crit}$ required for hydrogen burning. Show that the greater the mass of the star, the smaller the density at the point where $T_{\bf crit}$ is reached:**

$$\rho_{\bf crit} = 1.52\frac{1}{M^2}\left(\frac{k_B T_{\bf crit}}{\mu m_H G}\right)^3$$

   There are several ways to do this, but I recommend starting with the polytrope equations given in HKT (Eqs. 7.37-7.42). Perhaps the most straightforward way to do this is to consider the equation for central temperature (HKT Eq. 7.41, noting that Avogadro's number N_A is roughly the reciprocal of m_p):

$$T_{\rm crit} = \frac{1}{(n+1)(-\xi\theta')_{\xi_1}} \frac{G\mu m_p}{k_B} \frac{M}{R}$$

Plug in the expression for R as a function of $\bar{\rho} = \rho_{\rm crit}$, then solve for $\rho_{\rm crit}$:

$$T_{\rm crit} = \frac{1}{(n+1)(-\xi\theta')_{\xi_1}} \frac{G\mu m_p}{k_B} M \left(\frac{3M}{4\pi\rho_{\rm crit}}\right)$$

$$\rho_{\rm crit} = \frac{3}{4\pi M^2} \left(\frac{k_B T_{\rm crit}(n+1)(-\xi\theta')_{\xi_1}}{G\mu m_p}\right)^3$$

Plugging in numbers (from HKT Table 7.1), we find that $\boxed{\rho_{\rm crit} = 1.52\frac{1}{M^2}\left(\frac{k_B T_{\rm crit}}{\mu m_p G}\right)^3}$, as expected.

(c) **Noting the criterion for electron degeneracy, estimate the critical mass below which collapse is halted by electron degeneracy, not by hydrogen burning. After dropping factors of order unity, show that this mass is related to the Chandrasekhar limit, M_{Ch}, by the approximate relation**

$$\frac{M_{\bf crit}}{M_{Ch}} \sim \left(\frac{\mu_e}{\mu}\right)^{3/2} \left(\frac{k_B T_{\bf crit}}{m_e c^2}\right)^{3/4}$$

Evaluate this mass for $T_{\bf crit} = 5 \times 10^6$ K and $M_{Ch} = 1.4\ M_\odot$.

The criterion for electron degeneracy is that (degeneracy energy) > (thermal energy):

$$\frac{p_F^2}{2m_e} > k_B T$$

$$\frac{1}{2m_e}\left(\frac{3h^2 n_e}{8\pi}\right)^{2/3} > k_B T$$

Note that electron number density $n_e = \frac{\rho_{crit}}{\mu_e m_p}$, where $\rho_{\rm crit}$ is the critical density from the previous portion. Plugging these in and solving for $M_{\rm crit}$, we find:

$$M_{\rm crit} = \left(\frac{k_B T_{\rm crit}}{2m_e}\right)^{3/4} \left(\frac{(1.52)3}{8\pi\mu_e}\right)^{1/2} \left(\frac{h}{\mu G}\right)^{3/2} m_p^{-2}$$

Dividing this by the Chandrasekhar mass $M_{Ch} = \left(\frac{hc}{2\pi G}\right)^{3/2} \frac{1}{(\mu_e m_p)}^2$, we find

$$\frac{M_{\rm crit}}{M_{Ch}} = \left(\frac{1}{2}\right)^{3/4} \left(\frac{(1.52)3}{8\pi}\right)^{1/2} (2\pi)^{-3/2} \frac{\mu_e^{3/2}\mu^{-3/2}(k_B T_{\rm crit})^{3/4}}{(m_e c^2)^{3/4}}$$

Fortunately, all those numbers at the beginning are of order unity, so we can drop them:

$$\boxed{\frac{M_{\rm crit}}{M_{Ch}} \sim \left(\frac{\mu_e}{\mu}\right)^{3/2} \left(\frac{k_B T_{\rm crit}}{m_e c^2}\right)^{3/4}} \qquad (2)$$

For our object, $\mu = 0.62$ and μ_e is the number of baryons per electron. This is given by

$$\mu_e = \frac{1}{\sum \frac{\#\ \text{electrons}}{\text{baryon}} \times (\text{mass fraction})} = \frac{1}{X + \frac{1}{2}Y + \frac{1}{2}Z} = 1.18$$

Plugging in this value of μ along with the given values of $T_{\rm crit}$ and M_{Ch} into Equation (2), we find that $\boxed{M_{\rm crit} \sim 0.018\ M_\odot}$, which is approximately 20 Jupiter masses. Our object is a brown dwarf!

3. **Binary Stars**

(a) **Show that the minimum orbital period of the binary is**

$$P_{\mathbf{min}} \simeq 5\pi \left(\frac{15}{8\pi}\right)^{1/2} (G\rho)^{-1/2}$$

where ρ is the average stellar density. Evaluate $P_{\mathbf{min}}$ for a binary system of two red giants with $\rho = 10^{-6}$ g/cm^3, two Sun-like stars with $\rho = 1$ g/cm^3, two white dwarfs with $\rho = 10^6$ g/cm^3, and two neutron stars with $\rho = 3 \times 10^{14}$ g/cm^3.

From Kepler's laws:

$$\left(\frac{2\pi}{P}\right)^2 = \frac{GM_{\rm tot}}{a^3}$$

Substitute $a = a_{\rm min}$ using the expression given in the problem, and solve for $P = P_{\rm min}$:

$$P_{\rm min} = 2\pi \left(\frac{5}{2}\right)^{1/2} \left(\frac{M}{R^3}\right)^{-1/2} G^{-1/2}$$

Plug in the average stellar density $\rho = \frac{3}{4\pi}\frac{M}{R^3}$ to find

$$\boxed{P_{\rm min} = 5\pi \left(\frac{15}{8\pi}\right)^{1/2} (G\rho)^{-1/2}}$$

as expected. Evaluating this for binary systems with different densities:

- **2 red giants**: $\boxed{P_{\rm min} = 4.7 \times 10^7 \text{ s} = 544 \text{ days}}$
- **2 Sun-like stars**: $\boxed{P_{\rm min} = 4.7 \times 10^4 \text{ s} = 0.54 \text{ days}}$
- **2 white dwarfs**: $\boxed{P_{\rm min} = 47 \text{ s}}$
- **2 neutron stars**: $\boxed{P_{\rm min} = 2.7 \text{ ms}}$

(b) **Consider a red giant of $M_1 = 1\ M_\odot$, with a core mass $M_c = 0.5\ M_\odot$, envelope mass $M_e = 0.5\ M_\odot$, and radius $R_1 = 100\ R_\odot$. It undergoes a common-envelope event with a low-mass secondary star of mass M_2 and radius R_2, which ejects the envelope of the red giant. The α prescription for common-envelope events predicts the final orbital separation a_f:**

$$\alpha \left(\frac{GM_cM_2}{a_f} - \frac{GM_1M_2}{a_i}\right) = \frac{GM_cM_e}{R_1} \tag{3}$$

Solve this equation for a_f. Show that when α is of order unity and $M_2 \ll M_e$, the final orbital separation satisfies $a_f \ll a_i$, and this equation reduces to

$$a_f \simeq \frac{\alpha}{2}\frac{M_2}{M_e}R_1$$

First, solve Equation (3) for a_f:

$$\alpha \left(\frac{GM_cM_2}{a_f} - \frac{GM_1M_2}{a_i}\right) = \frac{GM_cM_e}{R_1}$$

$$\frac{M_cM_2}{a_f} = \frac{M_1M_2}{a_i} + \frac{M_cM_e}{\alpha R_1}$$

$$\boxed{a_f = \frac{M_cM_2}{2}\left(\frac{M_1M_2}{a_i} + \frac{M_cM_e}{\alpha R_1}\right)^{-1}}$$

Let $\alpha \sim 1$ and $M_2 \ll M_e$, so $a_i \simeq R_1$, and $M_1 M_2 \ll M_c M_e$. This means $\frac{M_1 M_2}{2a_i} \ll \frac{M_c M_e}{R_1}$, so the expression for a_f simplifies to

$$a_f \simeq \frac{M_c M_2}{2}\left(\cancel{\frac{M_1 M_2}{a_i}} + \frac{M_c M_e}{\alpha R_1}\right)^{-1}$$

$$\boxed{a_f \simeq \frac{\alpha}{2}\frac{M_2}{M_e} R_1}$$

Since $a_i \simeq R_1$, $a_f \simeq \frac{\alpha}{2}\frac{M_2}{M_e} a_i$. Then since $M_2 \ll M_e$, we find $a_f \ll a_i$.

(c) **A stellar merger will occur if the final separation a_f between the secondary and the primary's core is smaller than the minimum orbital separation possible for the secondary star. By replacing a_f with a_{min} for the secondary, and using $M_2 \ll M_c$, find the minimum secondary mass that can eject the envelope of the primary without merging with the core of the primary. Evaluate this mass for $\alpha = 0.5$ and typical brown dwarf radius $R_2 = 0.1\ R_\odot$.**

We *don't* want a stellar merger, so we require $a_f \geq a_{\rm min}$ for the secondary (which has R_2, M_2). Plug in the expression for a_f from the previous problem, and the expression for $a_{\rm min}$ given in the problem statement:

$$\frac{\alpha}{2}\frac{M_2}{M_e} R_1 \geq \frac{5}{2}\left(\frac{M_{\rm tot}}{M_2}\right)^{1/3} R_2$$

Solve this for M_2 to find

$$\boxed{M_2 \geq \left[\left(\frac{5}{\alpha}\right)^3 M_{\rm tot} M_e^3 \left(\frac{R_2}{R_1}\right)^3\right]^{1/4}}$$

Evaluate this for the given values (note that $M_1 = 1\ M_\odot$, and we can also make the approximation $M_{\rm tot} = M_1 + M_2 \approx M_1$) to find $\boxed{M_2 \geq 0.019\ M_\odot}$. This is about the mass of a brown dwarf! So brown dwarfs and low-mass stars can probably survive common-envelope evolution, but planets probably can't.

4. **Hydrogen Lines from Stars**

(a) **Consider a stellar atmosphere of pure hydrogen gas. Let's suppose H atoms only populate the $n = 1$ (ground) and $n = 2$ states. If n_2 is the number density of atoms with electrons in the $n = 2$ state, write down an expression for n_2/n_{tot}. You will need to use the Boltzmann factor in addition to your result from the Saha equation (Problem 3a of HW 3).**

The Boltzmann equation gives

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-E_{21}/k_B T} \tag{4}$$

To find the fraction of excited hydrogen atoms relative to total atoms $n_2/n_{\rm tot}$ (where $n_{\rm tot}$ includes both ionized and un-ionized hydrogen), we want

$$\frac{n_2}{n_{\rm tot}} = \frac{n_2}{n_H}\frac{n_H}{n_{\rm tot}} = \frac{n_2}{n_1 + n_2}\left(1 - \frac{n_e}{n_{\rm tot}}\right)$$

(Note that n_H is the number density of *un-ionized* hydrogen, and that the electron number density n_e should be equal to the number density of ionized hydrogen.)

We can rewrite this as

$$\boxed{\frac{n_2}{n_{\rm tot}} = \frac{1}{n_1/n_2 + 1}\left(1 - \frac{n_e}{n_{\rm tot}}\right)} \tag{5}$$

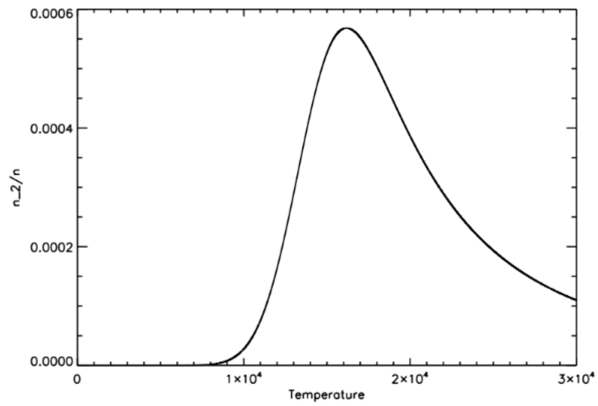

Figure 1: Plot of the n_2/n_{tot} as a function of temperature for a hydrogen gas.

(b) **If the continuum photosphere is at a total number density $n_{tot} = 10^{17}$ cm^{-3}, make a plot of n_2/n_{tot} as a function of stellar surface temperature. Recall that the energy levels of the H atom are given by $E = -13.6/n_2$ eV and the degeneracies are $g_n = 2n^2$. At what temperature does the value of n_2/n_{tot} peak? If the strength of Balmer lines is determined by the relative population n_2/n_{tot}, which stellar spectral type should show the most prominent H lines?**

We can solve Equation (5) using $n_e/n_{\rm tot}$ from the solution to Problem 3a in HW 3, and n_1/n_2 from Equation (4). Plugging in values, we can find $n_2/n_{\rm tot}$ as a function of temperature, shown in Figure 1. From this plot, $n_2/n_{\rm tot}$ peaks at $\boxed{T_{\rm eff} \approx 14,000 \text{ K}}$, which corresponds to the surface temperature of $\boxed{\text{B stars}}$.

(c) **The cross-section at line center for the production of Balmer lines is $\sigma \simeq 10^{-16}$ cm^2. Assuming an isothermal atmosphere for an A-type star with $g = 10^4$ cm s^{-2}, calculate the star's scale height. Then assume the value $n_2/n_{tot} = 10^{-4}$ is constant, and compute the optical depth at the center of the Balmer line at the continuum photosphere of an A-type star. Is this small or large? Calculate the density at which $\tau = 1$ near the center of the Balmer line. Are photons in the Balmer line emitted above or below the continuum photosphere of the star?**

Calculate the scale height $H = \frac{P}{g\rho}$. Assuming an ideal gas $P = \frac{\rho k_B T}{\mu m_p}$, we have $H = \frac{k_B T}{m_p g}$. Plugging in values (note that $T_{\rm eff} = 10,000$ K for an A-type star), we find $\boxed{H = 8.3 \times 10^7 \text{ cm}}$.

The optical depth is then given by $\tau = \alpha H$, where $\alpha = n_2\sigma$ is the absorption coefficient of Balmer photons (at line center). So $\tau = \frac{n_2}{n_{\rm tot}} n_{\rm tot}\sigma H$. Plug in the given values to find $\boxed{\tau = 8.3 \times 10^4}$. This is very large ($\tau \gg 1$), so the atmosphere at the continuum photosphere is opaque to Balmer photons!

Instead, consider when $\tau = 1$:

$$1 = \frac{n_2}{n_{\rm tot}} n_{\rm tot}\sigma H$$
$$n_{\rm tot} = \frac{1}{\frac{n_2}{n_{\rm tot}}\sigma H}$$

Substituting numbers, we find $\boxed{n_{\rm tot} = 1.2 \times 10^{12} \text{ cm}^{-3}}$.

This is a lower density than we originally assumed; the photons in the Balmer lines are emitted above the continuum photosphere of the star! To quantify how far above the continuum photosphere this is, note that the definition of atmospheric scale height is

$$n \propto e^{-z/H}$$

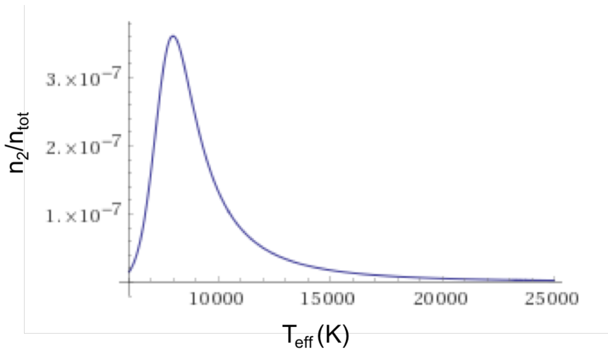

Figure 2: Plot of the n_2/n_{tot} as a function of temperature for a hydrogen gas.

where n is the atmospheric density and z is some height from the stellar "surface." Then the number of scale heights N_H between the continuum photosphere and where the Balmer photons are emitted is given by

$$\frac{n_{\text{continuum}}}{n_{\text{Balmer}}} = e^{N_H}$$

Solving this (using $n_{\text{continuum}} = 10^{17}$ cm^3 and $n_{\text{Balmer}} = 1.2 \times 10^{12}$ cm^3 that we just computed), we find that the Balmer photons are emitted $\boxed{N_H = 11.3}$ scale heights above the continuum photosphere.

(d) **Replot the value of n_2/n_{tot} as in part b, but with the number density you computed in part c. At which temperature do you now expect Balmer lines to be strongest?**

For density $n_{\text{tot}} = 1.2 \times 10^{12}$ cm^3, we replot n_2/n_{tot} in Figure 2. We find that n_2/n_{tot} now peaks at around $\boxed{T_{\text{eff}} \approx 8,000 \text{ K}}$.

Note that this is now lower than the effective temperature of an A star, primarily because of the incorrect value $n_2/n_{\text{tot}} = 10^{-4}$ we assumed in part (c).

In reality, with a better estimate of n_2/n_{tot}, one finds a density $n_{\text{tot}} \approx 10^{14}$ cm^{-3}, which yields a peak n_2/n_{tot} at $T_{\text{eff}} \approx 10,000$ K—the effective temperature of an A star! (This is why A stars are called A stars; the "Draper" classification system originally sorted stars in alphabetical order by the strength of their Balmer lines. The categories were then rearranged into their familiar OBAFGKM order when astronomers realized the different spectral classes corresponded to different temperatures.)