Ay123 Set 4 solutions

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1. **Cepheid variable**

   (a) **Use the continuity equation to show that a radial perturbation that satisfies $\Delta\rho/\rho = -\Delta V/V$ (where V is volume) implies that $\frac{\partial}{\partial r}\frac{\Delta r}{r} = 0$, where Δr is the radial Langrangian displacement.**

   From the continuity equation, you can get the mass equation:

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \tag{1}$$

   Perturb equation (1) with $r \to r + \Delta r$,$\rho \to \rho + \Delta\rho$:

$$\frac{\partial m}{\partial (r + \Delta r)} = 4\pi (r + \Delta r)^2 (\rho + \Delta\rho) \tag{2}$$

   The left hand side of equation (2) becomes

$$\begin{aligned}
\frac{\partial m}{\partial (r + \Delta r)} &= \frac{\partial m}{\partial [r(1 + \frac{\Delta r}{r})]} && (3)\\
&= \frac{\partial m}{\partial r(1 + \frac{\Delta r}{r}) + r\partial\left(\frac{\Delta r}{r}\right)} \qquad \text{(using chain rule)} && (4)\\
&= \frac{\partial m}{\partial r[1 + \frac{\Delta r}{r} + r\frac{\partial}{\partial r}(\frac{\Delta r}{r})]} && (5)\\
&= \frac{\partial m}{\partial r}\left(1 - \frac{\Delta r}{r} - r\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right)\right) && (6)
\end{aligned}$$

   Note that equation (6) comes from the fact that $\frac{\Delta r}{r}$ is small, so we can use the approximation $(1 + x)^{-1} \approx 1 - x$ for $x \ll 1$.

   The right hand side of equation (2) becomes (by expanding the expression and throwing out any non-linear perturbation terms):

$$\begin{aligned}
4\pi(r + \Delta r)^2(\rho + \Delta\rho) &= 4\pi(r^2 + 2r\Delta r + \cancel{(\Delta r)^2})(\rho + \Delta\rho) && (7)\\
&= 4\pi(r^2\rho + r^2\Delta\rho + 2r\rho\Delta r + \cancel{2r\Delta r\Delta\rho}) && (8)\\
&= 4\pi r^2\rho\left(1 + \frac{\Delta\rho}{\rho} + 2\frac{\Delta r}{r}\right) && (9)
\end{aligned}$$

   Now set the left- and right-hand sides (equations 6 and 9) equal:

$$\begin{aligned}
\frac{\cancel{\partial m}}{\cancel{\partial r}}\left(\cancel{1} - \frac{\Delta r}{r} - r\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right)\right) &= \cancel{4\pi r^2}\rho\left(\cancel{1} + \frac{\Delta\rho}{\rho} + 2\frac{\Delta r}{r}\right) && (10)\\
\Rightarrow \frac{\Delta\rho}{\rho} &= -3\frac{\Delta r}{r} - r\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right) && (11)
\end{aligned}$$

Using the fact that $\Delta\rho/\rho = -\Delta V/V$, equation (11) becomes

$$r\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right) = \frac{\Delta V}{V} - 3\frac{\Delta r}{r} \tag{12}$$

For a spherical star, $V = \frac{4}{3}\pi r^3$. So $\Delta V = \frac{4}{3}\pi(3r^2\Delta r)$, and $\frac{\Delta V}{V} = 3\frac{\Delta r}{r}$. Equation (11) then becomes $r\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right) = 0$. This must be true for all r, so we find

$$\boxed{\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right) = 0} \tag{13}$$

(b) **For a radial pulsation satisfying the equation derived in part (a), use the continuity equation to relate $\Delta\rho/\rho$ to $\Delta r/r$.**

From the previous part, we know that $\frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V} = -(3\frac{\Delta r}{r})$. So $\boxed{\frac{\Delta\rho}{\rho} = -3\frac{\Delta r}{r}}$.

(c) **Use this relation in the momentum equation to show that $\omega^2 = (3\gamma - 4)g/r$. What does this imply about the stability of the star when $\gamma < 4/3$?**

The perturbed momentum equation is

$$\rho\frac{\partial}{\partial t}\vec{v} = -\vec{\nabla}(\Delta P) + \frac{\Delta r}{r}\vec{\nabla}P - \Delta\rho\vec{g} - \rho\Delta\vec{g} \tag{14}$$

Rewrite this using $\vec{v} = \frac{\partial(\Delta\vec{r})}{\partial t}$ and divide both sides by ρr:

$$\frac{\Delta\ddot{r}}{r} = -\frac{1}{\rho r}\nabla\left(\frac{\Delta P}{P}P\right) + \frac{1}{r}\frac{\Delta r}{r}\frac{\nabla P}{\rho} - \frac{\Delta\rho}{\rho}\frac{g}{r} - \frac{\Delta g}{r} \tag{15}$$

Now we'll use a few substitutions:

- From part (b), we have $\boxed{\frac{\Delta\rho}{\rho} = -3\frac{\Delta r}{r}}$.
- Hydrostatic equilibrium yields $\nabla P = -\rho g$ (assuming pressure gradient is only in the radial direction), so $\frac{\nabla P}{\rho} = -g$. Then from an adiabatic EOS, we have $\frac{\Delta P}{P} = \gamma\frac{\Delta\rho}{\rho}$. So $\boxed{\frac{\Delta P}{P} = -3\gamma\frac{\Delta r}{r}}$. (Also, since $\frac{\partial}{\partial r}\left(\frac{\Delta P}{P}\right) = -3\gamma\frac{\partial}{\partial r}\left(\frac{\Delta r}{r}\right) = 0$ from part (a), $\frac{\Delta P}{P}$ is constant with respect to r. This means that we can pull $\frac{\Delta P}{P}$ out of the gradient.)
- From $g = \frac{GM}{r^2}$, we find that $\frac{\Delta g}{g} = -2\frac{\Delta r}{r} \Rightarrow \boxed{\Delta g = -2\frac{\Delta r}{r}g}$.

Plugging these into equation (15), we find

$$\frac{\Delta\ddot{r}}{r} = -\frac{\nabla P}{\rho}\frac{1}{r}\frac{\Delta P}{P} + \frac{1}{r}\frac{\Delta r}{r}\frac{\nabla P}{\rho} - \frac{\Delta\rho}{\rho}\frac{g}{r} - \frac{1}{r}\Delta g \tag{16}$$

$$\Delta\ddot{r} = -3\gamma\frac{g\Delta r}{r} - \frac{g}{r}\Delta r + 3\frac{g}{r}\Delta r + 2\frac{g}{r}\Delta r \tag{17}$$

$$= (-3\gamma + 4)\frac{g}{r}\Delta r \tag{18}$$

This has the form of

$$\Delta\ddot{r} = -\omega^2\Delta r \tag{19}$$

with $\boxed{\omega^2 = (3\gamma - 4)\frac{g}{r}}$ as expected. This implies that if $\gamma < 4/3$, $\omega^2 < 0$, so equation (19) has real exponential solutions $\Delta r \propto e^{\pm\omega t}$, and $\boxed{\text{the star is unstable}}$.

(d) **Using $\gamma = 5/3$, for a pulsation amplitude $\Delta r/r_0 = 0.1$, calculate the maximum surface velocity of the star in cgs units.**

For $\gamma = 5/3$, we find from the solution to the previous problem that $\omega^2 = \frac{g}{r}$. Then for periodic pulsations, $\Delta r = Ce^{\pm i\omega t}$, which has the real solution $\Delta r = C\cos\omega t$.

Solve for this equation by plugging in the maximum amplitude as an initial condition. At $t = 0$, $\Delta r = C = 0.1r_0$. So we have the final solution for the displacement: $\Delta r = 0.1r_0 C\cos\omega t$.

We can now use this to compute the velocity: $v = \frac{\mathrm{d}\Delta r}{\mathrm{d}t} = 0.1r_0\omega\sin\omega t$. The maximum velocity occurs when $\sin\omega t = 1$, so we have $v_{\max} = 0.1r_0\omega = 0.1r_0\sqrt{g/r_0}$. Plugging in values yields $\boxed{v_{\max} = 2.2\times 10^6 \text{ cm/s}}$.

(e) **Compute the fractional surface temperature perturbation $\Delta T_{\text{eff}}/T_{\text{eff}}$ and luminosity perturbation $\Delta L/L$.**

For adiabatic radial pulsations, we know $\frac{\Delta T}{T} = (\gamma-1)\frac{\Delta\rho}{\rho}$. Using the solution from part (b), this gives:

$$\frac{\Delta T}{T} = -3(\gamma-1)\frac{\Delta r}{r} \tag{20}$$

$$= -3(5/3-1)\frac{\Delta r}{r} \tag{21}$$

$$= -2\frac{\Delta r}{r} = -2(0.1) \tag{22}$$

So we have $\boxed{\frac{\Delta T_{\text{eff}}}{T_{\text{eff}}} = -0.2}$.

Also, we know $L = 4\pi R^2\sigma T^4$. This yields $\Delta L = 4\pi\sigma(2R\Delta R + 4T^3\Delta T)$, so:

$$\frac{\Delta L}{L} = \frac{(2R\Delta R + 4T^3\Delta T)}{R^2T^4} \tag{23}$$

$$= 2\left(\frac{\Delta R}{R}\right) + 4\left(\frac{\Delta T}{T}\right) = 2(0.1) + 4(-0.2) \tag{24}$$

So we have $\boxed{\frac{\Delta L}{L} = -0.6}$.

2. **Helioseismology**

(a) **Show that the sound speed close to the surface is given by $c_s^2 = (\gamma-1)gz$, where $z = R - r \ll R$ is the distance from the surface. Assume ρ, T, and P are zero at the surface.**

From hydrostatic equilibrium, we know $\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$. Rewrite this as

$$\frac{\mathrm{d}P}{\rho} = -g\mathrm{d}r \tag{25}$$

Assuming a polytropic equation of state, we have $P = K\rho^\gamma$, so $\rho = \left(\frac{P}{K}\right)^{1/\gamma}$. Then equation (25) becomes

$$K^{1/\gamma}P^{-1/\gamma}\mathrm{d}P = -g\mathrm{d}r \tag{26}$$

Integrate this to find

$$\int_{P(r)}^{P(R)} K^{1/\gamma}P^{-1/\gamma}\mathrm{d}P = -g\int_r^R \mathrm{d}r \tag{27}$$

$$K^{1/\gamma}\left(\frac{\gamma}{\gamma-1}\right)P^{\frac{\gamma-1}{\gamma}} = g(R-r) \tag{28}$$

Note that $K^{1/\gamma}P^{-1/\gamma} = \rho^{-1}$, and $z = R - r$, so we get:

$$\frac{P}{\rho} = \frac{\gamma-1}{\gamma}gz \tag{29}$$

Now use the definition of the sound speed $c_s^2 = \frac{\gamma P}{\rho}$ to find $\boxed{c_s^2 = (\gamma-1)gz}$.

(b) **For a horizontal wave number, $k_\perp$, and wave frequency, ω, find the maximum distance $z_{\mathbf{max}}$ that an acoustic wave can penetrate into the star. Write this result in terms of the associated spherical harmonic l. Do acoustic waves of higher frequency penetrate deeper or shallower into the star? What about waves of higher l?**

From the dispersion relation for acoustic waves:

$$\omega^2 = (k_r^2 + k_\perp^2)c_s^2 \tag{30}$$

$$k_r^2 = \frac{\omega^2}{c_s^2} - k_\perp^2 \tag{31}$$

For the wave to propagate, we need $k_r^2 > 0$:

$$\frac{\omega^2}{c_s^2} - k_\perp^2 > 0 \tag{32}$$

$$\frac{\omega^2}{c_s^2} > k_\perp^2 \tag{33}$$

Substitute the expression for c_s^2 from part (a) and the definition of the horizontal wavenumber $k_\perp = \frac{l(l+1}{r}$:

$$\frac{\omega^2}{(\gamma-1)gz} > \frac{l(l+1)}{r^2} \tag{34}$$

Solving for z, we find

$$\frac{\omega^2 r^2}{(\gamma-1)gl(l+1)} > z \tag{35}$$

Since $R - r \ll R$, we can just set $r \sim R$. Equation (35) then gives us a maximum z of

$$\boxed{z_{\max} = \frac{\omega^2 R^2}{(\gamma-1)gl(l+1)}} \tag{36}$$

Since $z_{\max}$ increases as ω increases, acoustic waves of higher frequency ω penetrate deeper into the star. However, at a given frequency ω, $z_{\max}$ decreases as l increases; waves of higher l penetrate shallower into the star.

(c) **Assume an oscillation mode has a radial displacement that is zero at $z = 0$ and $z = z_{\mathbf{max}}$. Show that**

$$\int_0^{z_{\mathbf{max}}} k_r \mathrm{d}z = (n+1)\pi$$

where k_r is the radial wavenumber and n is the number of nodes in the radial wavefunction.

Consider a series of standing waves in a chamber of length L, such that the radial displacement at $z = 0$ and $z = z_{\max} = 0$ is zero. A standing wave with $n = 0$ nodes will have $L = \lambda/2$, a standing wave with $n = 1$ nodes will have $L = \lambda$, a standing wave with $n = 2$ nodes will have $L = 3/2\lambda$, etc. This yields a pattern

$$L = \left(\frac{n+1}{2}\right)\lambda \text{ where } n = 0, 1, 2, ... \tag{37}$$

Using the definition of wavenumber ($k_r = \frac{2\pi}{\lambda}$), we can write

$$k_r L = \frac{2\pi}{\lambda}\left(\frac{n+1}{2}\right)\lambda \tag{38}$$

$$= \pi(n+1) \tag{39}$$

Then rewrite $L = \int_0^{z_{\max}} \mathrm{d}z$ to find

$$\boxed{\int_0^{z_{\max}} k_r \mathrm{d}z = (n+1)\pi} \tag{40}$$

(d) **From the equation derived in part (c), use the acoustic wave dispersion relation to find a relation between n, l, and ω.**

From equation (31), solve for k_r:

$$k_r = \left(\frac{\omega^2}{c_s^2} - k_\perp^2\right)^{1/2} \tag{41}$$

Then the left-hand side of equation (40) becomes

$$\int_0^{z_{\max}} k_r \mathrm{d}z = \int_0^{z_{\max}} \left(\frac{\omega^2}{c_s^2} - k_\perp^2\right)^{1/2} \mathrm{d}z \tag{42}$$

$$= \int_0^{z_{\max}} \left(\frac{\omega^2}{(\gamma-1)gz} - \frac{l(l+1)}{R^2}\right)^{1/2} \mathrm{d}z \tag{43}$$

where we have plugged in c_s^2 from part (a) and the definition of $k_\perp \equiv \frac{l(l+1)}{R^2}$.
Plugging in the definition of $z_{\max}$ from equation (36), this becomes

$$\int_0^{z_{\max}} k_r \mathrm{d}z = \frac{\sqrt{l(l+1)}}{R} \int_0^{z_{\max}} \left(\frac{\omega^2 R^2}{l(l+1)(\gamma-1)gz} - 1\right)^{1/2} \mathrm{d}z \tag{44}$$

$$= \frac{\sqrt{l(l+1)}}{R} \int_0^{z_{\max}} \left(\frac{z_{\max}}{z} - 1\right)^{1/2} \mathrm{d}z \tag{45}$$

To do this integral, change variables to $u = \frac{z}{z_{\max}}$, so $\mathrm{d}u = \frac{1}{z_{\max}}\mathrm{d}z$. Note the boundary conditions $u(z=0) = 0$ and $u(z = z_{\max}) = 1$.

$$\int_0^{z_{\max}} k_r \mathrm{d}z = \frac{\omega^2 R}{(\gamma-1)gl(l+1)} \frac{\sqrt{l(l+1)}}{R} \int_0^1 \sqrt{\frac{1}{u} - 1}\, \mathrm{d}u \tag{46}$$

Using your favorite solver of choice (mine's WolframAlpha), you can find that $\int_0^1 \sqrt{\frac{1}{u} - 1}\, \mathrm{d}u = \frac{\pi}{2}$.
Equation (40) then becomes

$$(n+1)\pi = \frac{\omega^2 R^2}{(\gamma-1)g\sqrt{l(l+1)}} \frac{\pi}{2} \tag{47}$$

$$\boxed{\omega^2 = 2(n+1)(\gamma-1)\sqrt{l(l+1)}\frac{g}{R}} \tag{48}$$

(e) **For $l = 0$, the appropriate value of $z_{\max}$ is R. Evaluate the equation derived in part (c) for $l = 0$ using the acoustic dispersion relation, still assuming g is constant. Show that the frequency spacing between modes of successive n is proportional to the square root of the density of the star.**

If $l = 0$, $k_\perp \equiv \frac{l(l+1)}{R^2} = 0$. So if we write equation (40) in the form of equation (43), we get

$$\int_0^{z_{\max}} \left(\frac{\omega^2}{(\gamma-1)gz}\right)^{1/2} \mathrm{d}z = (n+1)\pi \tag{49}$$

Plugging in $z_{\max} = R$ and integrating, we find

$$(n+1)\pi = \frac{\omega}{\sqrt{(\gamma-1)g}} \int_0^R z^{-1/2} \mathrm{d}z \tag{50}$$

$$= \frac{2\omega}{\sqrt{(\gamma-1)g}} R^{1/2} \tag{51}$$

Substitute $g = \frac{GM}{R^2}$ and solve for ω:

$$(n+1)\pi = \frac{2G^{-1/2}\omega_n}{\sqrt{\gamma-1}} M^{-1/2} R^{3/2} \tag{52}$$

$$\Rightarrow \omega = \frac{\sqrt{\gamma-1}}{2} G^{1/2}(n+1)\pi \left(\frac{M}{R^3}\right)^{1/2} \tag{53}$$

Then consider the frequency difference between mode n and $n+1$:

$$\omega_{n+1} - \omega_n = \frac{\sqrt{\gamma-1}}{2} G^{1/2}((n+2)-(n+1))\pi \left(\frac{M}{R^3}\right)^{1/2} \tag{54}$$

$$= \frac{\sqrt{\gamma-1}}{2} G^{1/2}\pi \left(\frac{M}{R^3}\right)^{1/2} \tag{55}$$

Note that $\frac{\sqrt{\gamma-1}}{2} G^{1/2}\pi$ is constant.

Then since density $\rho \propto \frac{M}{R^3}$, we find that $\boxed{(\omega_{n+1} - \omega_n) \propto \rho^{1/2}}$ as expected.

3. **Nuclear cross sections**

   (a) **To simplify the integral given in the problem text, write it in terms of E and b, where $E = m_{AB}v^2/2$ and**

$$b = \frac{\sqrt{2m_{AB}}\pi Z_A Z_B e^2}{\hbar} = 0.99 Z_A Z_B \sqrt{m_{AB}} (\mathbf{MeV})^{1/2}$$

The starting equation:

$$\langle \sigma v \rangle = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v \frac{S}{E} \exp\left(-\frac{mv^2}{2k_B T}\right) \exp\left(-\frac{2\pi Z_A Z_B e^2}{\hbar v}\right) v^2 \mathrm{d}v \tag{56}$$

As suggested, define b and E:

$$b = \frac{\sqrt{2m_{AB}}\pi Z_A Z_B e^2}{\hbar} = 0.99 Z_A Z_B \sqrt{m_{AB}} (\mathrm{MeV})^{1/2} \tag{57}$$

$$E = \frac{mv^2}{2} \Leftrightarrow v = \sqrt{\frac{2E}{m}} \Leftrightarrow \mathrm{d}v = \sqrt{\frac{2}{m}} \frac{1}{2} \frac{dE}{E^{1/2}} \tag{58}$$

and assume $S(E) \approx S_0$. Then equation (52) becomes

$$\langle \sigma v \rangle = \frac{8\pi}{m^2} \left(\frac{m}{2\pi k_B T}\right)^{3/2} S_0 \int_0^\infty \exp\left(-\frac{E}{k_B T}\right) \exp\left(-\frac{b}{E^{1/2}}\right) \mathrm{d}E \tag{59}$$

$$\boxed{\langle \sigma v \rangle = \sqrt{\frac{8}{\pi m}} \frac{1}{(k_B T)^{3/2}} S_0 \int_0^\infty \exp\left(-\frac{E}{k_B T}\right) \exp\left(-\frac{b}{E^{1/2}}\right) \mathrm{d}E} \tag{60}$$

   (b) **Recast the integral as the integral of a Gaussian. The Gaussian is centered at E_0 with a width Δ and an amplitude C. Find the values of C, E_0, and Δ, and do the integral.**

Note that a Gaussian can be expressed as:

$$g(E) = C \exp\left(-\frac{(E-E_0)^2}{(\Delta/2)^2}\right) \tag{61}$$

The integrand in equation (56) is not a Gaussian but can be approximated as one. The peak of a Gaussian occurs where the derivative is zero:

$$\frac{\mathrm{d}}{\mathrm{d}E}\left[\exp\left(-\frac{E}{k_BT}\right)\exp\left(-\frac{b}{E^{1/2}}\right)\right] = 0 \text{ at } E = E_0 \tag{62}$$

$$\Rightarrow \left(-\frac{1}{k_BT} + \frac{b}{2E_0^{3/2}}\right)\left[\exp\left(-\frac{E}{k_BT}\right)\exp\left(-\frac{b}{E^{1/2}}\right)\right] = 0 \tag{63}$$

Since $e^x \neq 0$ at all x, the first term in equation (59) must be zero, and we can solve for E_0:

$$\frac{b}{E_0^{3/2}} = \frac{2}{k_BT} \Rightarrow \boxed{E_0 = \left(\frac{bk_BT}{2}\right)^{2/3}} \tag{64}$$

In order for $\langle\sigma v\rangle$ to be approximated as a Gaussian, the peak value at $E = E_0$ must be the same for both $g(E)$ and $\langle\sigma v\rangle$:

$$\sqrt{\frac{8}{\pi m}}\frac{1}{(k_BT)^{3/2}}S_0\exp\left(-\frac{E_0}{k_BT}\right)\exp\left(-\frac{b}{E_0^{1/2}}\right) = C\exp\left(-\frac{(E_0-E_0)^2}{(\Delta/2)^2}\right) \tag{65}$$

This gives us C:

$$\boxed{C = \sqrt{\frac{8}{\pi m}}\frac{1}{(k_BT)^{3/2}}S_0\exp\left(-\frac{E_0}{k_BT}\right)\exp\left(-\frac{b}{E_0^{1/2}}\right)} \tag{66}$$

Finally, to find Δ, note that the second derivative of $g(E)$ should be equal to the second derivative of $\langle\sigma v\rangle$ at the peak:

$$\frac{\partial^2}{\partial E^2}\left[C\exp\left(-\frac{(E-E_0)}{k_BT} - b\left(\frac{1}{E^{1/2}} - \frac{1}{E_0^{1/2}}\right)\right)\right] = \frac{\partial^2}{\partial E^2}\left[C\mathrm{e}^{-\left(\frac{2(E-E_0)}{\Delta}\right)^2}\right] \tag{67}$$

$$\frac{\partial}{\partial E}\left[\left(-\frac{1}{k_BT} + \frac{b}{2E^{3/2}}\right)\mathrm{e}^{-\frac{(E-E_0)}{k_BT} - b\left(\frac{1}{E^{1/2}} - \frac{1}{E_0^{1/2}}\right)}\right] = \frac{\partial}{\partial E}\left[\left(\frac{8(E-E_0)}{\Delta^2}\right)\mathrm{e}^{-\left(\frac{2(E-E_0)}{\Delta}\right)^2}\right] \tag{68}$$

$$\left[\left(-\frac{1}{k_BT} + \frac{b}{2E^{3/2}}\right)^2 - \frac{3b}{4E^{5/2}}\right]\mathrm{e}^{-\frac{(E-E_0)}{k_BT} - b\left(\frac{1}{E^{1/2}} - \frac{1}{E_0^{1/2}}\right)} = \left[\left(\frac{8(E-E_0)}{\Delta^2}\right)^2 - \frac{8}{\Delta^2}\right]\mathrm{e}^{-\left(\frac{2(E-E_0)}{\Delta}\right)^2} \tag{69}$$

Evaluate equation (65) at the peak $E = E_0$ and solve for Δ:

$$-\frac{3b}{4E_0^{5/2}} = -\frac{8}{\Delta^2} \tag{70}$$

$$\Delta^2 = \frac{32}{3}\frac{E_0^{5/2}}{b} = \frac{32}{3b}\left(\frac{bk_BT}{2}\right)^{5/3} \tag{71}$$

$$\Rightarrow \Delta = \frac{4}{\sqrt{3}}\frac{b^{1/3}}{2^{1/3}}(k_BT)^{5/6} \tag{72}$$

Plug in $b = \frac{2}{k_BT}E_0^{3/2}$ to find

$$\boxed{\Delta = \sqrt{\frac{16E_0k_BT}{3}}} \tag{73}$$

Now that we have C, E_0, and Δ, we can do the integral in equation (57):

$$\langle \sigma v \rangle \approx C \int_0^\infty \exp\left(-\frac{4(E-E_0)^2}{\Delta^2}\right) \mathrm{d}E \tag{74}$$

Using the integral for a Gaussian and plugging in the definitions for C and Δ, we find

$$\begin{aligned}
\langle \sigma v \rangle &\approx C \frac{\sqrt{\pi}}{2} \frac{\Delta}{2} &(75)\\
&\approx \frac{\pi}{4} \sqrt{\frac{2}{\pi m}} \left(\frac{1}{k_B T}\right)^{3/2} S_0 \mathrm{e}^{-\frac{E_0}{k_B T}} \mathrm{e}^{-\frac{b}{E_0^{1/2}}} \sqrt{\frac{16 E_0 k_B T}{3}} &(76)\\
&\approx \sqrt{\frac{8}{3m}} \frac{E_0^{1/2}}{k_B T} S_0 \mathrm{e}^{-\frac{E_0}{k_B T}} \mathrm{e}^{-\frac{b}{E_0^{1/2}}} &(77)
\end{aligned}$$

Plug in $E_0^{1/2} = \left(\frac{b k_B T}{2}\right)^{1/3} = \left[\frac{\sqrt{m}\pi Z_A Z_B e^2 k_B T}{\sqrt{2}\hbar}\right]^{1/3}$ to find

$$\langle \sigma v \rangle \approx \sqrt{\frac{8}{3m}} \left(\frac{\sqrt{m}\pi e^2}{\sqrt{2}\hbar}\right)^{1/3} \frac{Z_A^{1/3} Z_B^{1/3}}{(k_B T)^{2/3}} S_0 \mathrm{e}^{-\frac{\left(\frac{b k_B T}{2}\right)^{2/3}}{k_B T}} \mathrm{e}^{-\frac{b}{\left(\frac{b k_B T}{2}\right)^{1/3}}} \tag{78}$$

$$\boxed{\langle \sigma v \rangle \approx \left[\frac{2^{4/3}\pi^{1/3} e^{2/3}}{m^{1/3}\hbar^{1/3}}\right] \frac{Z_A^{1/3} Z_B^{1/3}}{(k_B T)^{2/3}} S_0 \mathrm{e}^{-\frac{b^{2/3}}{(k_B T)^{1/3}}\left(\frac{3}{2^{2/3}}\right)}} \tag{79}$$

(c) **Derive the expression**

$$\langle \sigma v \rangle \propto \frac{1}{Z_A Z_B m_{AB}} S_0 \tau^2 \mathbf{e}^{-\tau} \tag{80}$$

and give the expression for τ.

Start with the solution from the previous section and rewrite in terms of b:

$$\begin{aligned}
\langle \sigma v \rangle &\propto \left(\frac{e^2 Z_A Z_B}{m\hbar (k_B T)^2}\right)^{1/3} S_0 \mathrm{e}^{-\frac{3b^{2/3}}{2^{2/3}(k_B T)^{1/3}}} &(81)\\
&\propto \frac{1}{m^{1/2}} \left(\frac{b}{(k_B T)}\right)^{1/3} S_0 \mathrm{e}^{-\frac{3b^{2/3}}{2^{2/3}(k_B T)^{1/3}}} &(82)
\end{aligned}$$

Now use $b = \frac{2E_0^{3/2}}{k_B T}$ to rewrite equation (78) in terms of E_0:

$$\begin{aligned}
\langle \sigma v \rangle &\propto \left[\frac{E_0^{3/2}}{(k_B T)^3}\right]^{1/3} \frac{S_0}{m^{1/2}} \exp\left(-\frac{3E_0}{k_B T}\right) &(83)\\
&\propto \frac{k_B T}{E_0^{3/2}} \left(\frac{E_0}{k_B T}\right)^2 \frac{S_0}{m^{1/2}} \exp\left(-\frac{3E_0}{k_B T}\right) &(84)
\end{aligned}$$

Motivated by the form of equation (80), define $\boxed{\tau = \frac{3E_0}{k_B T}}$ and rewrite this equation:

$$\langle \sigma v \rangle \propto \frac{k_B T}{m^{1/2} b k_B T} S_0 \tau^2 \mathrm{e}^{-\tau} \tag{85}$$

Note that $b \propto \sqrt{m} Z_A Z_B$, so equation (81) becomes

$$\boxed{\langle \sigma v \rangle \propto \frac{1}{Z_A Z_B m_{AB}} S_0 \tau^2 \mathrm{e}^{-\tau}} \tag{86}$$

(d) **The nuclear energy generation rate scales with temperature as $\epsilon = \epsilon_0 \rho T^\eta$. Find η for the reactions:**

  i. $p + p$ **at** $T = 1.0 \times 10^7$ **K and** 1.5×10^7 **K**

  ii. ^{7}**Be** $+ p$ **at** $T = 1.5 \times 10^7$ **K**

  iii. ^{14}**N** $+ p$ **at** $T = 1.5 \times 10^7$ **K and** 2.5×10^7 **K**

The equation for η is given in KWW Equation 18.39, but we can review how it's derived. Assume the energy generation rate goes as the cross section:

$$\langle \sigma v \rangle \propto \epsilon = \epsilon_0 \rho T^\eta \tag{87}$$
$$\tau^2 \mathrm{e}^{-\tau} \propto T^\eta \Rightarrow \tau^2 \mathrm{e}^{-\tau} = A T^\eta \tag{88}$$

To solve for η, take the derivative with respect to T:

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[\tau^2 \mathrm{e}^{-\tau}\right] = A\eta \frac{T^\eta}{T} \tag{89}$$
$$\left[2\tau \frac{\mathrm{d}\tau}{\mathrm{d}T} - \tau^2 \frac{\mathrm{d}\tau}{\mathrm{d}T}\right] \mathrm{e}^{-\tau} = \frac{\eta}{T} A T^\eta \tag{90}$$
$$\frac{2}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}T} - \frac{\mathrm{d}\tau}{\mathrm{d}T} = \frac{\eta}{T} \tag{91}$$
$$\eta = T\frac{\mathrm{d}\tau}{\mathrm{d}T}\left(\frac{2}{\tau} - 1\right) \tag{92}$$

Solve for $\frac{\mathrm{d}\tau}{\mathrm{d}T}$ using the definition of $\tau = \frac{3E_0}{k_B T} = \frac{3b^{2/3}}{2^{2/3}(k_B T)^{1/3}}$. This yields $\frac{\mathrm{d}\tau}{\mathrm{d}T} = -\frac{1}{3}\frac{\tau}{T}$:

$$\eta = -\frac{1}{3}(2 - \tau) \tag{93}$$
$$= \frac{\tau}{3} - \frac{2}{3} \tag{94}$$
$$= \frac{E_0}{k_B T} - \frac{2}{3} \tag{95}$$
$$= \left[\frac{\sqrt{2m}\pi Z_A Z_B e^2}{2\hbar}\right]^{2/3} \frac{1}{(k_B T)^{1/3}} - \frac{2}{3} \tag{96}$$

Plugging in numbers for the given equations:

  i. $m = \frac{m_A m_B}{m_A + m_B} = \frac{m_p}{2}$ and $Z_A = Z_B = 1$: $\boxed{\eta = 4.2}$ at $T = 10^7$ K, $\boxed{\eta = 3.6}$ at $T = 1.5 \times 10^7$ K

  ii. $m = \frac{7}{8} m_p$ and $Z_A = 4$, $Z_B = 1$: $\boxed{\eta = 12.4}$ at $T = 1.5 \times 10^7$ K

  iii. $m = \frac{14}{15} m_p$ and $Z_A = 7$, $Z_B = 1$: $\boxed{\eta = 19}$ at $T = 1.5 \times 10^7$ K, $\boxed{\eta = 16}$ at $T = 2.5 \times 10^7$ K

4. **Deuterium burning**

(a) **What are the values of m_{AB}, Z_A, Z_B, and b^2 (defined in the previous problem) for Deuterium burning? Write down the resulting thermally-averaged value of $\langle \sigma v \rangle$ for D fusion.**

For deuterium burning, $\boxed{Z_A = Z_B = 1}$. The reduced mass m_{AB} is given by $m_{AB} = \frac{m_p m_d}{m_p + m_d} = \frac{2m_p^2}{m_p + 2m_p} \Rightarrow \boxed{m_{AB} = \frac{2}{3} m_p}$.

Then using the formula from part (3b), we find $b = 0.99(1)(1)\sqrt{2/3}(\mathrm{MeV})^{1/2} \Rightarrow \boxed{b^2 = 0.66 \text{ MeV}}$ Plugging in the given value of S_0 and our value for b we get

$$\boxed{\langle \sigma v \rangle = 8 \times 10^4 \text{ barn} \cdot \text{ cm s}^{-1}\ \mathrm{T}_7^{-2/3} \mathrm{e}^{-17.3 \mathrm{T}_7^{-1/3}}} \tag{97}$$

(b) **The luminosity of a protostar on the Hayashi track is**

$$L \approx 0.2 L_\odot \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^2$$

By equating this with the energy released by Kelvin-Helmholtz contraction, calculate the local contraction time t_c as a function of the mass and radius of the star. Does the contraction time get shorter or longer as the star contracts?

The energy released by Kelvin-Helmholtz contraction is simply the gravitational binding energy of a spherical object (note that the exact coefficient of 3/5 in front doesn't matter much, just the main scaling relations):

$$E_{KH} = -U_{gr} = \frac{3}{5}\frac{GM^2}{R} \tag{98}$$

The local contraction time is then given by dividing E_{KH} by the luminosity (energy generation rate):

$$t_c = \frac{E_{KH}}{L} \tag{99}$$

$$= \frac{3}{5}\frac{GM^2}{R}\left(\frac{M_\odot}{M}\right)^{1/2}\left(\frac{R_\odot}{R}\right)^2 \frac{1}{0.2 L_\odot} \tag{100}$$

$$\boxed{t_c = 9 \times 10^7 \text{ y} \left(\frac{M}{M_\odot}\right)^{3/2}\left(\frac{R}{R_\odot}\right)^{-3}} \tag{101}$$

As the star contracts, the radius decreases and the mass stays the same, so $\boxed{t_c \text{ gets longer}}$.

(c) **What is the lifetime t_D of a D nucleus at the center of the star in terms of the local density and temperature? Use the properties of $n = 3/2$ polytropes to write t_D as a function of M and R. Does the D lifetime get shorter or longer as the star contracts?**

The lifetime of a D nucleus is just the inverse of the collision rate

$$t_D^{-1} = n_H \langle \sigma v \rangle \tag{102}$$

Substituting $n_H = \rho/\mu m_p$ and $\langle \sigma v \rangle$ from equation (93), we can get t_D in terms of temperature and density:

$$t_D = 1.04 \times 10^{-5} \rho^{-1} T_7^{2/3} e^{17.3 T_7^{-1/3}} \tag{103}$$

$$\boxed{t_D = 2.24 \times 10^{-10} \rho^{-1} T^{2/3} e^{3730 T^{-1/3}}} \tag{104}$$

At the center of the sun, for an $n = 3/2$ polytrope we have

$$\rho_c = a_n \langle \rho \rangle = \frac{5.99 \cdot 3M}{4\pi R^3} = 1.43 \frac{M}{R^3} \tag{105}$$

The pressure is given by $P_c = 0.77 GM^2/R^4$ (cf. HKT Eq. 7.39) so the temperature is

$$T_c = \frac{\mu m_p P_c}{\rho_c k} \tag{106}$$

$$= \frac{.6 m_p .77 G}{1.43 k}\frac{M^2}{R^4}\frac{R^3}{M} \tag{107}$$

$$= 2.6 \times 10^{-16} \frac{M}{R} \tag{108}$$

We can use these expressions to write t_D in terms of M and R only:

$$t_D = 6.4\times10^{-21}\left(\frac{M}{R^3}\right)^{-1}\left(\frac{M}{R}\right)^{2/3} e^{5.8\times10^8(M/R)^{-1/3}} \tag{109}$$

$$= 6.4\times10^{-21} M^{-1/3} R^{7/3} e^{5.8\times10^8(M/R)^{-1/3}} \tag{110}$$

$$= \boxed{10^{-6}\left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{R}{R_\odot}\right)^{7/3} e^{19\left(\frac{M_\odot}{M}\frac{R}{R_\odot}\right)^{1/3}}\ \text{s}} \tag{111}$$

$$\tag{112}$$

As R decreases t_D will decrease; $\boxed{\text{the D lifetime gets shorter}}$ as the star contracts.

(d) **Give the numerical value of R_D for $M = 0.03$ and $0.1M_\odot$. For each of these two cases, also determine the central temperature of the star T_c and the D lifetime t_D when $R = R_D$. Does D fusion occur before or after the star reaches the main sequence?**

To find the critical radius we set $t_c(R, M) = t_D(R, M)$:

$$9\times10^7\ \text{y}\left(\frac{M}{M_\odot}\right)^{3/2}\left(\frac{R}{R_\odot}\right)^{-3} = 10^{-6}\ \text{s}\left(\frac{\text{M}}{\text{M}_\odot}\right)^{-1/3}\left(\frac{\text{R}}{\text{R}_\odot}\right)^{7/3} \text{e}^{19\left(\frac{\text{M}_\odot}{\text{M}}\frac{\text{R}}{\text{R}_\odot}\right)^{1/3}} \tag{113}$$

$$\left(\frac{M}{M_\odot}\right)^{11/6}\left(\frac{R}{R_\odot}\right)^{-16/3} = 5.8\times10^{-22} e^{19\left(\frac{M_\odot}{M}\frac{R}{R_\odot}\right)^{1/3}} \tag{114}$$

This expression is a function of only M and R, so for a given M we have an equation for $R = R_D$. For $M = .03M_\odot$ we get

$$\left(\frac{R}{R_\odot}\right)^{-16/3} = 3.6\times10^{-19} e^{61\left(\frac{R}{R_\odot}\right)^{1/3}} \Rightarrow R/R_\odot = 0.45 \tag{115}$$

Using our expression for T_c (eq. 104) at $M = .03M_\odot$ and $R = .45R_\odot$ we find $\boxed{T_c = 5.0\times10^5\ \text{K}}$. Plugging into t_c (or equivalently t_D), we find the time is $\boxed{5\times10^6\ \text{y}}$.

Repeat the same steps for $M = .1M_\odot$ and find

$$\left(\frac{R}{R_\odot}\right)^{-16/3} = 4.0\times10^{-20} e^{41\left(\frac{R}{R_\odot}\right)^{1/3}} \Rightarrow R/R_\odot = 1.21 \tag{116}$$

Using our expression for T_c at $M = .1M_\odot$ and $R = 1.21R_\odot$ we find $\boxed{T_c = 6.1\times10^5\ \text{K}}$. Plugging into t_c (or equivalently t_D), we find the time is $\boxed{1.6\times10^6\ \text{y}}$.

D fusion occurs at lower temperatures than H fusion. Therefore, it occurs $\boxed{\text{before}}$ the star reaches the main sequence.

(e) **Explain quantitatively whether D fusion can halt (at least temporarily) the KH contraction of the star. If so, how long does the "D main sequence" last for the two cases considered in part (d) above?**

D fusion can halt KH contraction when the luminosity from D fusion overcomes the convective luminosity $L_{conv} \approx 0.2L_\odot\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{R}{R_\odot}\right)^2$. The "D main sequence" will last as long as Deuterium fusion can power the star:

$$t_{D\ m.s.} \sim \frac{E_D}{L_{conv}}, \tag{117}$$

where $E_D \sim N_D \cdot \chi$ is the energy released from D fusion. Here, χ is the energy liberated per reaction, which for D fusion is 5.5 MeV $= 8.8\times10^{-6}$ erg. The number of Deuterium atoms can be found using $N_D = 2 \times 10^{-5} N_H = 2 \times 10^{-5} M/m_p$. Solving for $t_{D\ m.s.}$ gives

$$\begin{aligned} t_{D\ m.s.} &= \frac{N_D \cdot \chi}{L} \\ &= 3.2 \times 10^{14} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{-2} \\ &= 10^7 \text{ y} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{-2} \end{aligned}$$

Plugging in for our two cases:

$M = .03 M_\odot$ and $R = .45 R_\odot \Rightarrow \boxed{t_{D\ m.s.} = 8.6 \times 10^6 \text{ y}}$

$M = .1 M_\odot$ and $R = 1.23 R_\odot \Rightarrow \boxed{t_{D\ m.s.} = 2.1 \times 10^6 \text{ y}}$

If $t_{D\ m.s.}$ is longer than t_c, then there is at least a temporary halt in the contraction, and indeed this is the case.

Now that you have finished this problem you might find it interesting to look at some of the figures in Burrows et al., 2001, RVMP, 73, 719, which show evolutionary calculations of the contraction of low mass stars and brown dwarfs and the effects of D fusion. After D fusion comes Li fusion and then, if the mass is big enough, H fusion.