Ay123 Set 3 solutions

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October 23 2018

1. **Equation of state and the Chandrasekhar mass**

   (a) **Using the Fermi-Dirac distribution for non-relativistic electrons, derive the relationship between density and pressure, and hence appropriate value of γ and K.**
   The Fermi-Dirac distribution is given by

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \mathrm{d}p\ p^2 \left[\exp\left(\frac{\epsilon-\mu}{k_B T}\right)+1\right]^{-1} \tag{1}$$

   Then pressure is given by

$$P_e = \frac{1}{3}\int vp\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p}\mathrm{d}^3p \tag{2}$$

$$= \frac{8\pi}{3h^3}\int_0^\infty \mathrm{d}p\ vp^3 \left[\exp\left(\frac{\epsilon-\mu}{k_B T}\right)+1\right]^{-1} \tag{3}$$

   We can rewrite this in terms of momentum using the fact that the Fermi-Dirac distribution is essentially a step function:

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} \mathrm{d}p\ vp^3 \tag{4}$$

   where $p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$ is the Fermi momentum. In the non-relativistic limit, substitute $v \sim p/m_e$ and the definition of the Fermi momentum into equation (4):

$$P_e = \frac{8\pi}{3h^3 m_e}\int_0^{p_F} \mathrm{d}p\ p^4 \tag{5}$$

$$= \frac{8\pi}{3h^3 m_e}\frac{p_F^5}{5} \tag{6}$$

   Note that electron number density $n_e = \frac{\rho}{\mu_e m_p}$, where $\mu_e = 2$ for ^{4}He. Then we can write the electron pressure as

$$\boxed{P_e = \left(\frac{8\pi}{3h^3}\right)^{-2/3}\left(\frac{1}{5m_e}\right)\left(\frac{1}{\mu_e m_p}\right)^{5/3}\rho^{5/3}} \tag{7}$$

   Then $\boxed{\gamma = 5/3}$ and $\boxed{K = \left(\frac{8\pi}{3h^3}\right)^{-2/3}\left(\frac{1}{5m_e}\right)\left(\frac{1}{\mu_e m_p}\right)^{5/3} = 3.135\times10^{12}\ \mathrm{cm^4 g^{-2/3} s^{-2}}}$ in the polytropic equation $P = K\rho^\gamma$.

   (b) **Using the mass-radius relations we derived for polytropes, derive the mass-radius relation for a white dwarf. Calculate the radius of a 1 $\mathbf{M_\odot}$ white dwarf.**
   From the discussion of polytropes (HKT Eq. 7.40):

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta_n)^{n-1}}\right]_{\xi_1}^{1/n}\frac{G}{n+1}M^{1-1/n}R^{-1+3/n} \tag{8}$$

Here, $n = \frac{1}{\gamma-1} = 3/2$ for a $\gamma = 5/3$ polytrope. From HKT Table 7.1, $\xi_1 = 3.6538$ and $-\theta_n(\xi_1) = 0.20330$ for $n = 3/2$. Set this equal to the equation for K from part (a) and solve for R: $\boxed{R = (1.11 \times 10^{20} \text{ cm g}^{1/3}) \text{ M}^{-1/3}}$.

For a 1 $\text{M}_\odot$ white dwarf, this yields $\boxed{R = 8.81 \times 10^8 \text{ cm} \approx 0.01 \text{ R}_\odot}$.

(c) **Now assume the helium white dwarf is supported by highly relativistic degeneracy pressure. Use the Fermi-Dirac distribution to derive the appropriate values of γ and K, and then derive its mass. Express the mass in units of $\text{M}_\odot$.**

Now consider relativistic electrons, so $v = c$. Then

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} \mathrm{d}p \; vp^3 \tag{9}$$

$$= \frac{8\pi c}{3h^3} \int_0^{p_F} \mathrm{d}p \; p^3 \tag{10}$$

$$= \frac{8\pi c}{3h^3} \frac{p_F^4}{4} \tag{11}$$

$$= \frac{8\pi c}{3h^3} \frac{c}{4} \left(\frac{3h^3 n_e}{8\pi} \right)^{4/3} \tag{12}$$

Again, plugging in the definition for $n_e = \frac{\rho}{\mu_e m_p}$, we find that the pressure goes as

$$P_e = \left(\frac{8\pi}{3h^3} \right)^{-1/3} \left(\frac{c}{4} \right) \left(\frac{1}{\mu_e m_p} \right)^{4/3} \rho^{4/3} \tag{13}$$

So $\boxed{\gamma = 4/3}$ and $\boxed{K = \left(\frac{8\pi}{3h^3}\right)^{-1/3} \left(\frac{c}{4}\right) \left(\frac{1}{\mu_e m_p}\right)^{4/3} = 4.9 \times 10^{14} \; g^{-1/3} \text{ cm}^3 \text{ s}^{-2}}$ (again assuming $\mu_e = 2$ for ^{4}He).

To get the mass, start with the equation given in class for a polytrope:

$$M = 4\pi \left(\frac{n+1}{4\pi G} K \right)^{3/2} \rho_c \frac{3-n}{2n} \left(-\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi} \right)_{\xi_1} \tag{14}$$

where $n = \frac{1}{\gamma-1} = 3$ for a $\gamma = 4/3$ polytrope. Then

$$M = 4\pi \left(\frac{K}{\pi G} \right)^{3/2} (\xi_1)^2 (-\theta_n)_{\xi_1} \tag{15}$$

Use values from HKT Table 7.1: $\xi_1 = 6.8969$ and $-\theta_n(\xi_1) = -.04243$ for $n = 3$ to find $\boxed{M = 1.43 \text{ M}_\odot}$. This is very close to the Chandrasekhar mass!

(d) **Recalculate this mass for a relativistic white dwarf made of pure gold (^{197}Au). Gold is currently worth \$39.47/g. What is the value of this golden dwarf?**

The only thing that changes here is the value of μ_e, which is the number of baryons per electron. For ^{197}Au, which has atomic number $Z = 79$, this yields $\mu_e = \frac{197}{79} = 2.49$.

Then use the formula from part (c) to find $K = 3.7 \times 10^{14} \text{ g}^{-1/3} \text{ cm}^3 \text{ s}^{-2}$. Plug this into equation (15) for M to find $\boxed{M = 1.88 \times 10^{33} \text{ g} = 0.94 \text{ M}_\odot}$.

The cost of this golden dwarf is $\boxed{\$7.42 \times 10^{34}}$!

(e) **If the golden dwarf has a radius of 3000 km and an internal temperature of 10^7 K, will most of its mass be liquid gold or solid gold?**

To determine if the gold is solid (crystallized) or not, calculate the ratio of Coulomb to thermal energy $\Gamma_c = \frac{Z^2 e^2}{a k_B T}$.

The separation between gold ions is given by $a = \left(\frac{3}{4\pi n_{ion}}\right)^{1/3}$, where n_{ion} is the number density of ions. The number density is given by $n_{ion} = \frac{3M}{197m_p(4\pi R^3)}$.

This yields

$$\Gamma_c = Z^2e^2\left(\frac{3M}{197m_p(4\pi R^3)}\right)^{1/3}\left(\frac{3}{4\pi}\right)^{-1/3}(k_BT)^{-1} \tag{16}$$

Plugging in values gives $\Gamma_c = 6.3 \times 10^3 \gg 100$, which is well into the regime of crystallization. Most of the mass of the white dwarf will be $\boxed{\text{solid}}$ gold.

(f) **If the surface layers of the golden dwarf have an opacity of $\kappa = 10$ cm^2/g, what is the surface pressure at the photosphere? If the gold at the photosphere can be treated as an ideal gas and has a temperature of 10^4 K, what is its density? The radius of a gold atom is $a_{Au} = 1.74 \times 10^{-8}$ cm. Is the gold at the surface likely to be pressure ionized?**

Recall (from, e.g., the last HW set) that the surface pressure at the photosphere is given by $P(\tau_s) = \frac{2g_s}{3\kappa}$. Plugging in the surface gravity $g_s = \frac{GM}{R^2}$ yields $P(\tau_s) = \frac{2GM}{3R^2\kappa}$.

Use $\kappa_s = 10$ cm^2/g and the values of M and R from the previous parts to find the surface pressure at the photosphere $\boxed{P(\tau_s) = 9.3 \times 10^7 \text{ erg cm}^{-3}}$.

Now, assuming the photosphere is an ideal gas, $P = \frac{\rho k_B T}{\mu m_p}$ so $\rho = \frac{P\mu m_p}{k_B T}$. Recall that μ is the mean molecular weight $\mu = \frac{\text{\# baryons}}{\text{\# particles}} = \frac{197}{1+79} = 2.46$. Plug in values to find $\boxed{\rho = 2.8 \times 10^{-4} \text{ g/cm}^3}$.

Pressure ionization occurs when $\rho \gtrsim \frac{m_N}{4\pi a_N^3/3}$. For ^{197}Au, $m_N = 197m_p$ and $a_N = 1.74 \times 10^{-8}$ cm. This yields $\frac{m_N}{4\pi a_N^3/3} = 14.9$ g/cm^3. So $\rho < \frac{m_N}{4\pi a_N^3/3}$, and pressure ionization is unlikely.

2. **Nuclides and kilonova event rates**

(a) **Consult the table of nuclides. Identify two stable, pure s-process nuclides and two stable, pure r-process nuclides.**

Note that the nuclides must be stable (or at least long-lived)!

Some examples of pure s-process elements: ^{92}Mo, ^{102}Pd

Some examples of pure r-process elements: ^{76}Ge, ^{94}Zr

(b) **In the Sun, the mass fraction of r-process nuclides is $X_{rp} \sim 10^{-7}$. The stellar mass of the Milky Way is $M_{\mathbf{MW}} \sim 10^{11}$ M$_\odot$. Assuming similar abundances in other stars, estimate the total r-process mass within the Milky Way.**

$M_{rp,\text{MW}} = X_{rp}M_{\text{MW}} = \boxed{10^4 \text{ M}_\odot}$.

(c) **Assuming r-process elements are produced solely in neutron star mergers, and that $M_{rp} \sim 0.03$ M$_\odot$ of r-process nuclides were expelled from GW170817, estimate the number of neutron star mergers that have occurred in the Milky Way, and the average neutron star merger rate over the $\tau_{\mathbf{MW}} \sim 10$ Gyr lifetime of the Milky Way.**

The number of neutron star mergers is given by N_{NSM}=(mass of r-process elements)/(mass of r-process elements per NSM). So $N_{\text{NSM}} = \frac{M_{rp,\text{MW}}}{M_{rp}} = \boxed{3.3 \times 10^5}$.

The average rate of neutron star mergers is just $N_{\text{NSM}}/\tau_{\text{MW}} = \boxed{3.3 \times 10^{-5} \text{ yr}^{-1}}$.

(d) **Type Ia supernovae synthesize $M_{\mathbf{Fe}} \sim 0.5$ M$_\odot$ of iron, whose mass fraction in the Sun is $X_{\mathbf{Fe}} \sim 10^{-3}$. Estimate the average Type Ia supernovae rate of the Milky Way.**

As in part (b), first compute the total mass of iron in the Milky Way: $M_{\text{Fe,MW}} = 10^8$ M$_\odot$. Then follow the procedure in part (c) to compute the number of Type Ia SNe: $N_{\text{Ia}} = \frac{M_{\text{Fe,MW}}}{M_{\text{Fe}}} = 2 \times 10^8$. Divide this by the lifetime of the Milky Way to get the average Type Ia SNe rate: $N_{\text{Ia}}/\tau_{\text{MW}} = \boxed{0.02 \text{ yr}^{-1}}$.

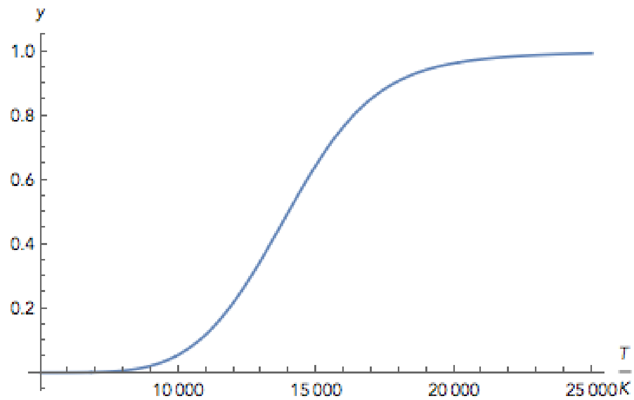

Figure 1: Plot of the ionization fraction n_e/n_{tot} as a function of temperature for a pure hydrogen gas.

(e) **Estimate the number of neutron star mergers that have occurred in a dwarf galaxy with metallicity $\sim 10^{-1}$ that of the Milky Way, and stellar mass $M_{\mathbf{gal}} \sim 10^6$ $\mathbf{M}_\odot$. Do we expect all dwarf galaxies to be enriched in r-process elements such as Europium?**

As in part (b), first compute the total mass of r-process elements in the dwarf galaxy: $M_{rp,\text{gal}} = X_{rp} M_{\text{gal}} = 10^{-2}$ $\text{M}_\odot$.

Then as in part (c), compute the number of neutron star mergers: $N_{\text{NSM}} = \frac{M_{rp,\text{gal}}}{M_{rp}} = \boxed{0.33}$. We do not expect all dwarf galaxies to be enriched in r-process elements.

3. **Free electrons at low temperatures**

(a) **Use the Saha equation to calculate the fraction of free electrons n_e/n_{tot} for a pure hydrogen gas. Plot your result as function of temperature. Assume $n_{tot} = 10^{17}$ cm^{-3} as is appropriate for the solar photosphere, where n_{tot} is the total number of nuclei (ionized + neutral).**

The Saha equation is

$$\frac{n_e n_{H+}}{n_{H^0}} = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\frac{\chi_H}{k_B T}} \tag{17}$$

We want $y = n_e/(n_{H+} + n_{H^0})$, which is simply the degree of ionization. The Saha equation can then be written as

$$\frac{y^2}{1-y} = \frac{1}{n_{tot}} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\frac{\chi_H}{k_B T}} \tag{18}$$

This can then be solved numerically for y as a function of T using the quadratic formula (note that y must be positive, so use the positive solution). The result is shown in Figure 1.

(b) **Calculate the fraction of free electrons $n_e/n_{tot,H}$ as a function of temperature for the combined hydrogen + metal gas. As in (a), assume $n_{tot,H} = 10^{17}$ cm^{-3}. Make a plot showing the ratio of n_e for part (b) to n_e for part (a) as a function of temperature. Below what temperature does the assumption of a pure hydrogen gas start to become inaccurate for predicting n_e?**

Using

$$\lambda \equiv \left(\frac{h^2}{2\pi m_e kT}\right)^{1/2}$$

we can write Saha for hydrogen (rewriting equation 17):

$$\frac{n_H}{n_{H+} n_e} = \lambda^3 e^{\chi_1/kT} \tag{19}$$

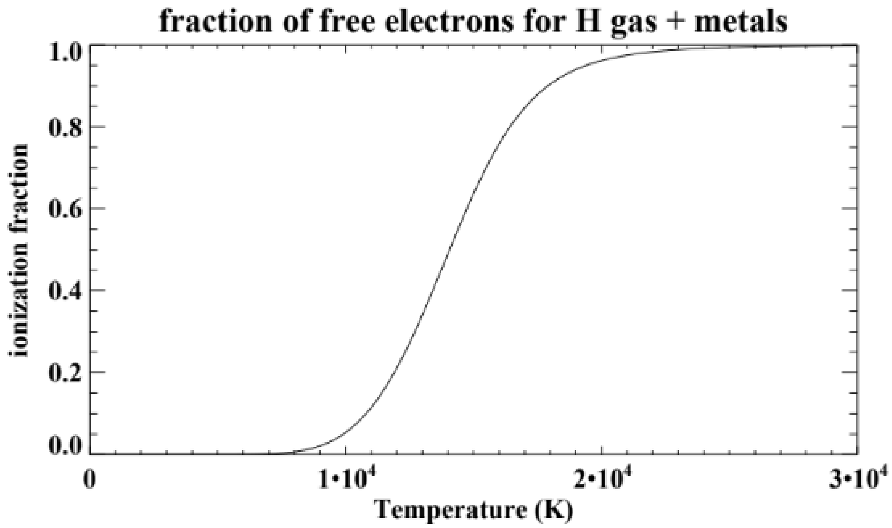

Figure 2: Plot of the ionization fraction $n_e/n_{tot,H}$ as a function of temperature for a metal and hydrogen gas.

For the metal we have

$$\frac{n_m}{n_{m+}n_e} = \lambda^3 e^{\chi_m/kT} \tag{20}$$

We also have the relations

$$n_{tot,H} = n_{H+} + n_H = 10^{17} \tag{21}$$

$$n_{tot,m} = n_{m+} + n_H = 10^{11} \tag{22}$$

and

$$n_{H+} + n_{m+} = n_e \tag{23}$$

Using 20 and 22 we can solve for n_{m+} in terms of n_e only:

$$\frac{n_m}{n_m+} = \frac{n_{tot,m}}{n_{m+}} - 1 = n_e \lambda^3 e^{\chi_m/kT}$$

$$\rightarrow n_{m+} = \frac{n_{tot,m}}{n_e \lambda^4 e^{\chi_m/kT}}$$

We can likewise use (2) and (4) to solve for n_{H+}:

$$n_{H+} = \frac{n_{tot,H}}{n_e \lambda^4 e^{\chi_1/kT}}$$

Therefore, we can write down an expression for the electron density by (6):

$$n_e = \frac{n_{tot,m}}{n_e \lambda^4 e^{\chi_m/kT}} + \frac{n_{tot,H}}{n_e \lambda^4 e^{\chi_1/kT}}$$

We use $n_e = n_{tot,H} y$ (where y is the ionization fraction we want) to rewrite this as a function of y:

$$y = \frac{n_{tot,m}}{n_{tot,H}} \frac{1}{y n_{tot,H} \lambda^3 e^{\chi_m/kT} + 1} + \frac{1}{y n_{tot,H} \lambda^4 e^{\chi_1/kT} + 1}$$

I couldn't be bothered to actually solve this, but the person who created Figure 2 used IDL's *FZ_ROOTS*.

The ratio of the ionization fractions is shown in Figure 3. This shows that metals are important at $T < 4500$ K.

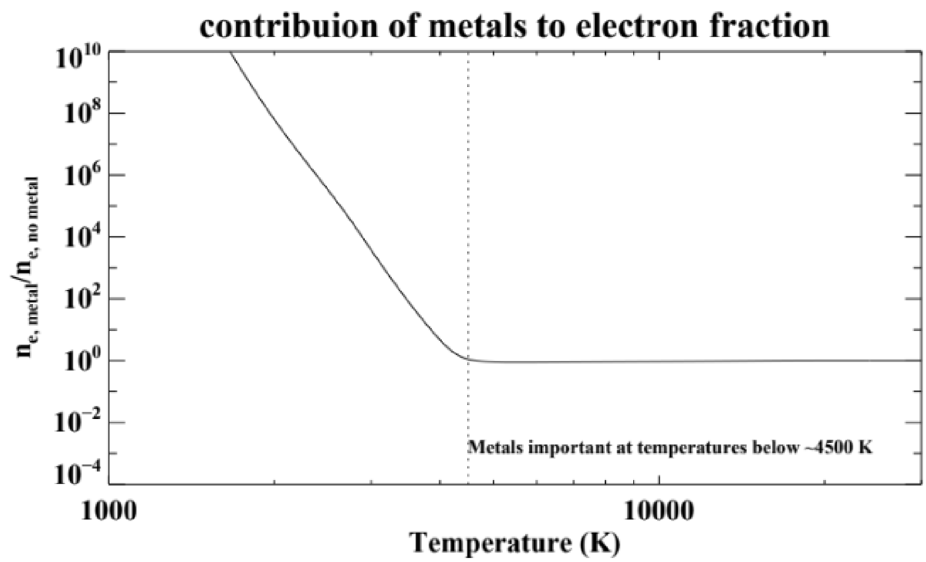

Figure 3: Ratio of ionization fractions for metal+hydrogen gas and pure hydrogen gas.