Ay123 Set 2 solutions

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1. **Convection**

   (a) **If a convective blob accelerates from zero velocity over a mixing length Λ according to $\frac{\mathrm{d}}{\mathrm{d}r}v_{con} = |N|$, find the maximum convective velocity v_{con} in terms of N and the mixing length Λ. Express v_{con} in terms of α, γ, $(\nabla_{ad} - \nabla)$, ρ, and c_s.**

   We want to solve the differential equation $\frac{\mathrm{d}}{\mathrm{d}r}v_{con} = |N|$ to get $v_{con}(r)$. This is pretty straightforward, since we assume N is constant over a mixing length, so we find:

$$v_{con}(r) = |N|r + C \tag{1}$$

   To solve for the integration constant, we use the boundary condition $v_{con}(r = 0) = 0$. This conveniently yields $C = 0$.

   Now we want the maximum v_{con}, which should occur at the mixing length $r = \Lambda$. The maximum convective velocity should therefore be $\boxed{v_{con} = |N|\Lambda}$. We can also rewrite this by plugging in $\Lambda = \alpha H$, $N^2 = \frac{g}{H}(\nabla_{ad} - \nabla)$, and $H = P/(\rho g)$:

$$\begin{aligned} v_{con} &= |N|\Lambda & (2)\\ &= \left(\frac{g}{H}\right)^{1/2}(\nabla_{ad} - \nabla)^{1/2}\alpha H & (3)\\ &= \left(\frac{P}{\rho}\right)^{1/2}(\nabla_{ad} - \nabla)^{1/2}\alpha & (4) \end{aligned}$$

   Recall that a polytropic equation of state is given by $P = \rho^\gamma$, and the sound speed is given by $c_s^2 = \gamma P/\rho$. So $(P/\rho)^{1/2} = \sqrt{\rho^{\gamma-1}}$, and the sound speed is $c_s = \sqrt{\gamma\rho^{\gamma-1}}$.

   Plugging this into equation (4) yields $\boxed{v_{con} = \gamma^{-1/2}c_s(\nabla_{ad} - \nabla)^{1/2}\alpha}$.

   (b) **Express the kinetic energy flux $F_{con} = \rho v_{con}^3$ of upgoing convective blobs.**

   Substituting v_{con} from (a), the convective flux is $\boxed{F_{con} = \rho\gamma^{-3/2}(\alpha c_s)^3(\nabla_{ad} - \nabla)^{3/2}}$.

   (c) **For $\alpha = 2$, what is the value of $(\nabla_{ad} - \nabla)$ required for convection to carry the Sun's luminosity? What is the corresponding maximum convective velocity v_{con}, and how does this compare to the sound speed c_s?**

   The energy flux going through the base of the convective zone r is $F = \frac{L_\odot}{4\pi r^2}$. Setting this equal to the convective energy flux F_{con} from part (b), we can solve for $(\nabla_{ad} - \nabla)$:

$$\begin{aligned} \rho\gamma^{-3/2}(\alpha c_s)^3(\nabla_{ad} - \nabla)^{3/2} &= \frac{L_\odot}{4\pi r^2} & (5)\\ (\nabla_{ad} - \nabla) &= \left(\frac{L_\odot}{\rho 4\pi r^2}\right)^{2/3}\gamma(\alpha c_s)^{-2} & (6) \end{aligned}$$

   Substituting the given values (note that $\gamma = 5/3$ for an ideal gas and $L_\odot = 4 \times 10^{33}$ erg/s is a good value to remember), we find $\boxed{(\nabla_{ad} - \nabla) = 1.2 \times 10^{-7}}$.

   Using the expression from part (a), this corresponds to $\boxed{v_{con} = 1.1 \times 10^4 \text{ cm/s}}$, which is much slower than the sound speed $c_s = 2 \times 10^7$ cm/s.

(d) **Assuming convection carries all the Sun's luminosity, use the expression for N^2 to find the density gradient in a convection zone.**

From the given equation for N^2, we know that

$$g\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}+\frac{g}{c_s^2}\right]=-N^2 \tag{7}$$

We can solve this for $\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}$. Plugging in $N^2=\frac{g}{H}(\nabla_{ad}-\nabla)$, $H=P/(\rho g)$, and equation (6) for $(\nabla_{ad}-\nabla)$, we find

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} = \frac{-N^2}{g}-\frac{g}{c_s^2} \tag{8}$$

$$= -\frac{1}{H}(\nabla_{ad}-\nabla)-\frac{g}{c_s^2} \tag{9}$$

$$= -\frac{\rho g}{P}\left(\frac{L_\odot}{4\pi r^2\rho}\right)^{2/3}\frac{\gamma}{\alpha^2 c_s^2}-\frac{g}{c_s^2} \tag{10}$$

Now remember that we're using a polytropic equation of state, so $c_s^2=\gamma P/\rho$. Substituting this into equation (10), we find

$$\boxed{\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}=-\frac{g}{c_s^2}\left[\left(\frac{L_\odot}{4\pi r^2\rho c_s^3}\right)^{2/3}\frac{\gamma^2}{\alpha^2}+1\right]} \tag{11}$$

as expected.

For the solar values given in part (c), we find that $\boxed{\left(\frac{L_\odot}{4\pi r^2\rho c_s^3}\right)^{2/3}\frac{\gamma^2}{\alpha^2}=2\times10^{-7}}$, which is much smaller than 1. The first term in equation (11) can then be treated as negligible, so the density gradient becomes $\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}=\frac{g}{c_s^2}$. This suggests that the Sun's density profile is not strongly dependent on α at the convective zone.

(e) **At the Sun's surface, what is the value of $(\nabla_{ad}-\nabla)$ required to carry the Sun's luminosity? What is the corresponding convective velocity, and how does it compare to the sound speed? Will $\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}$ depend on α near the surface of the Sun?**

As in part (c), plug the given values into equation (6) to find $\boxed{(\nabla_{ad}-\nabla)=0.49}$. This corresponds to $\boxed{v_{con}=8.7\times10^5\ \text{cm/s}}$, which is larger than the sound speed $c_s=8\times10^5$ cm/s. This suggests that convection cannot carry the Sun's luminosity at the Sun's surface.

As in part (d), compute the first term in brackets in equation (11) using the given values. This yields $\boxed{\left(\frac{L_\odot}{4\pi r^2\rho c_s^3}\right)^{2/3}\frac{\gamma^2}{\alpha^2}=0.8}$. This is comparable with 1, so the term depending on α in equation (11) is not negligible. The Sun's density profile therefore depends on α near the surface of the Sun.

2. **Fully convective cool stars**

(a) **Assuming a star's envelope is convective and composed of an ideal gas with polytropic index γ, find the temperature within the envelope.**

Because the star is marginally unstable to convection, we can use the convection criterion $\nabla=\nabla_{ad}$. Using the definitions for ∇ and ∇_{ad}, we can write this as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \frac{\gamma-1}{\gamma} \tag{12}$$

$$\frac{P}{T}\frac{\mathrm{d}T/\mathrm{d}r}{\mathrm{d}P/\mathrm{d}r} = \frac{\gamma-1}{\gamma} \tag{13}$$

Hydrostatic equilibrium gives us $\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm\rho}{r^2}$. (Note that we can leave $m = m(r)$ in this equation, or we can just recognize that the mass of the envelope is negligible to the core mass M, so just set $m(r) = M$). Also, assuming an ideal gas gives us $P = \frac{\rho k_B T}{\mu m_H}$. Plugging these into equation (13) yields

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{\gamma - 1}{\gamma}\frac{T\rho}{P}\left(-\frac{GM}{r^2}\right) \tag{14}$$

$$= -\frac{\gamma - 1}{\gamma}\frac{\mu m_p}{k_B}\frac{GM}{r^2} \tag{15}$$

Now let's integrate to find $T = T(r)$:

$$\int_R^r \frac{\mathrm{d}T}{\mathrm{d}r}\mathrm{d}r = \frac{\gamma - 1}{\gamma}\frac{\mu m_p}{k_B}GM\int_R^r -\frac{\mathrm{d}r}{r^2} \tag{16}$$

$$T(r) - T(R) = \frac{\gamma - 1}{\gamma}\frac{\mu m_p}{k_B}GM\left(\frac{1}{r} - \frac{1}{R}\right) \tag{17}$$

Note that $T(R)$ is just the surface temperature T_s, so we find that

$$\boxed{T(r) = \frac{\gamma - 1}{\gamma}\frac{\mu m_p}{k_B}GM\left(\frac{1}{r} - \frac{1}{R}\right) + T_s} \tag{18}$$

(b) **The polytropic convective envelope extends nearly all the way to the photosphere. Use this to derive a scaling between pressure and temperature at the photosphere.**

The gas is a polytrope, so $P = K\rho^\gamma$. We want to get rid of ρ in this expression by writing it in terms of P and T. The envelope is an ideal gas, so $P = \frac{\rho k_B T}{\mu m_H}$. This means that $\rho = \frac{P\mu m_H}{k_B T} \propto PT^{-1}$. We can plug this scaling into the polytropic equation to write P as a function of T:

$$P \propto P^\gamma T^{-\gamma} \tag{19}$$

$$P^{1-\gamma} \propto T^{-\gamma} \tag{20}$$

$$\boxed{P \propto T^{\frac{\gamma}{\gamma-1}}} \tag{21}$$

(c) **Assuming nearly constant opacity above the photosphere, show that photospheric pressure is $P_s = \frac{2g_s}{3\kappa_s}$, where κ_s is the photospheric opacity.**

Start from hydrostatic equilibrium and integrate from the surface to infinity:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho \tag{22}$$

$$\int_R^\infty \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}r = -\int_R^\infty g\rho\mathrm{d}r \tag{23}$$

$$P(r \to \infty) - P(R) = -g\int_R^\infty \rho\mathrm{d}r \tag{24}$$

Since the total mass of the star is dominated by the mass of the core M, g is roughly constant and can be pulled out of the integral.

Then use the definition of optical depth $\tau = \int_R^\infty \kappa\rho\mathrm{d}r$. Since we're assuming nearly constant opacity $\kappa = \kappa_s$ above the photosphere, this yields

$$\tau = \kappa_s\int_R^\infty \rho\mathrm{d}r \tag{25}$$

Plug equation (25) into equation (24), noting that $P(r \to \infty) = 0$ and $P(R)$ is the surface pressure P_s:

$$P_s = \tau\frac{g_s}{\kappa_s} \tag{26}$$

Finally, the photospheric optical depth is $\tau = 2/3$. Substituting this into equation (26) yields $\boxed{P_s = \frac{2g_s}{3\kappa_s}}$.

(d) **Use the fact that H- opacity dominates in cool stars and has the scaling $\kappa_{H-} \propto T^9$, and results from parts (b) and (c), to determine a scaling between M, R, and T_{eff} for cool stars.**

Set the results of parts (b) and (c) equal and plug in $g \propto MR^{-2}$ and $\kappa \propto T^9$. Keep the proportionality constant in the polytropic equation of state, since it turns out this constant (we'll call it K, as in $P = KT^{\frac{\gamma}{\gamma-1}}$) depends on M and R:

$$KT^{\frac{\gamma}{\gamma-1}} \propto \frac{g_s}{\kappa_s} \tag{27}$$

$$KT^{\frac{\gamma}{\gamma-1}} \propto MR^{-2}T^{-9} \tag{28}$$

$$MR^{-2} \propto KT^{9+\frac{\gamma}{\gamma-1}} \tag{29}$$

Unfortunately, K also depends on M and R, so we have to solve for it somehow. This is given in HKT p.336, but the easiest way is to consider the central pressure and temperature. The central pressure is roughly given by hydrostatic equilibrium:

$$\frac{\mathrm{d}P}{\mathrm{d}r} \propto \frac{M\rho}{r^2} \tag{30}$$

$$\frac{P_c}{R} \propto \frac{M}{R^2}\frac{M}{R^3} \tag{31}$$

$$P_c \propto M^2R^{-4} \tag{32}$$

The central temperature is given by the ideal gas law:

$$T_c \propto \frac{P_c}{\rho} \tag{33}$$

$$T_c \propto MR^{-1} \tag{34}$$

As we know from last week's homework, the polytropic equation of state should also hold in the center of the star, so we can solve for K using $P_c = KT_c^{\frac{\gamma}{\gamma-1}}$. This yields $K = M^{-1/2}R^{-3/2}$.

Now we can plug this expression for K into equation (30) to get $\boxed{M^{3/2}R^{-1/2} \propto T^{9+\frac{\gamma}{\gamma-1}}}$. For an ideal gas of $\gamma = 5/3$, this is $\boxed{M^{3/2}R^{-1/2} \propto T^{23/2}}$.

(e) **Draw an evolutionary track for a red giant branch star on an HR diagram.**

Relate L, M, and T using the fact that for a blackbody, $L \propto R^2T^4$. First relate L, M, and R using the solution from part (d):

$$T \propto L^{1/4}R^{-1/2} \Rightarrow M^{3/2}R^{-1/2} \propto L^{23/8}R^{-23/4} \Rightarrow L^{23/8} \propto M^{3/2}R^{21/4} \tag{35}$$

Then get rid of R using $R \propto L^{1/2}T^{-2}$:

$$L^{23/8} \propto M^{3/2}L^{21/8}T^{-21/2} \Rightarrow L^{1/4}M^{3/2} \propto T^{21/2} \tag{36}$$

Since M is fixed for a given star, this yields $\boxed{L \propto T^{42}}$, a near vertical track on the HR diagram.

(f) **Main sequence G/K/M stars have $L \propto M^4$. Add the lower part of the main sequence to the HR diagram.**

Instead of treating M fixed, we'll plug the scaling relation for $M \propto L^{1/4}$ into equation (36) to find $\boxed{L^{5/8} \propto T^{21/2}}$.

3. **Hot massive stars**

Consider a family of stars in which the opacity is dominated by Thomson (electron) scattering and in which nuclear energy is generated by the CNO cycle.

(a) **How do opacity and energy generation depend on density and temperature?**

Electron scattering opacity goes as $\boxed{\kappa = \kappa_0}$ (is independent of ρ and T).

Energy generation from the CNO cycle scales as $\boxed{\epsilon = \epsilon_0 \rho T^{17}}$.

(b) **Find the relation between radius and mass. Also find the relation between luminosity and mass.**

Most of this was done in class, but in case you want a review of homology relations, here's the full derivation anyway. Start with the equations of stellar structure, plugging in the expressions for ϵ and κ from part (a):

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \tag{37}$$

$$\frac{\mathrm{d}r}{\mathrm{d}m} = -\frac{1}{4\pi r^4 \rho} \tag{38}$$

$$\frac{\mathrm{d}L}{\mathrm{d}m} = \epsilon_0 \rho T^{17} \tag{39}$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = -\frac{3\kappa_0 L}{64\pi^2 a c r^4 T^3} \tag{40}$$

Rewrite the dependent variables using an internal mass coordinate $m = m_* \left(\frac{M}{M_*}\right)$:

$$r = r_* \left(\frac{M}{M_*}\right)^a \tag{41}$$

$$\rho = \rho_* \left(\frac{M}{M_*}\right)^b \tag{42}$$

$$L = L_* \left(\frac{M}{M_*}\right)^c \tag{43}$$

$$T = T_* \left(\frac{M}{M_*}\right)^d \tag{44}$$

$$P = P_* \left(\frac{M}{M_*}\right)^e \tag{45}$$

Now rewrite the equations of stellar structure (38-41) using these new variables. Note that these equations must hold at all masses, so we can set the exponents equal to each other:

$$\left(\frac{M}{M_*}\right)^{e-1} \frac{\mathrm{d}P_*}{\mathrm{d}m_*} = -\left(\frac{M}{M_*}\right)^{1-4a} \frac{Gm_*}{4\pi r_*^4} \qquad \Rightarrow 4a + e = 2 \tag{46}$$

$$\left(\frac{M}{M_*}\right)^{a-1} \frac{\mathrm{d}r_*}{\mathrm{d}m_*} = -\left(\frac{M}{M_*}\right)^{-b-2a} \frac{1}{4\pi r_*^4 \rho_*} \qquad \Rightarrow 3a + b = 1 \tag{47}$$

$$\left(\frac{M}{M_*}\right)^{c-1} \frac{\mathrm{d}L_*}{\mathrm{d}m_*} = \left(\frac{M}{M_*}\right)^{b+17d} \epsilon_0 \rho_* T_*^{17} \qquad \Rightarrow c = 1 + b + 17d \tag{48}$$

$$\left(\frac{M}{M_*}\right)^{d-1} \frac{\mathrm{d}T_*}{\mathrm{d}m_*} = -\left(\frac{M}{M_*}\right)^{c-4a-3d} \frac{3\kappa_0 L_*}{64\pi^2 a c r_*^4 T_*^3} \qquad \Rightarrow 4d = c - 4a + 1 \tag{49}$$

Also assume ideal gas pressure support $P = \frac{\rho k_B T}{\mu m_p}$:

$$\left(\frac{M}{M_*}\right)^{e} P_* = \left(\frac{M}{M_*}\right)^{b+d} \frac{\rho_* k_B T_*}{\mu m_p} \qquad \Rightarrow e = b + d \tag{50}$$

Write this as a matrix equation:

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -17 & 0 \\ 4 & 0 & -1 & 4 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \tag{51}$$

Now, just invert this matrix to solve for (a, b, c, d, e).

Using the solutions from class, we find $(a, b, c, d, e) = (4/5, -7/5, 3, 1/5, -6/5$. Plugging these into equations (42) and (44), we get the mass-radius relation $\boxed{R \propto M^{4/5}}$ and the mass-luminosity relation $\boxed{L \propto M^3}$. (Note that this could also be done using radiation pressure instead of gas pressure; this would yield $R \propto M^{19/40}$ and $L \propto M$.)

(c) **Where is this family of stars on the Hertzsprung–Russell diagram?**

As we know from class, these are high-mass stars with $M = 10 - 100 M_\odot$. You might already know that these are on the $\boxed{\text{upper main sequence}}$ (since they're hot enough to have electron scattering, but still H-burning with the CNO cycle) in the upper left part of the HR diagram.

But we can also figure out where they go by finding a relation between L and T. For a blackbody we know $L = 4\pi R^2 \sigma T^4$, so $T \propto L^{1/4} R^{-1/2}$. Plugging in the relations from part (b), we find that $T \propto M^{0.35}$. Since $L \propto M^3$, we can combine these relations and find that $L \propto T^{8.57}$. If you plug in a couple of numbers (using the Sun as reference), you can find that they're on the upper part of the main sequence.

(d) **Generate MESA models. Record their radius, luminosity, and temperature early on the main sequence, and compare to your above results.**

MESA should give something like:

$M/M_\odot$	$\log(R/R_\odot)$	$T_{\rm eff}$ (K)	$\log(L/L_\odot)$
10	0.649	24420	3.803
20	0.820	33490	4.694
30	0.921	38470	5.137
40	0.997	41430	5.417
60	1.109	44560	5.767

This is pretty close to what we derived in part (b), though not quite exact.

(e) **Consider a universe where electrons are twice as massive. How would this affect the mean molecular weight of stars? How would this affect the electron scattering opacity? Derive a scaling relation between stellar radius, mass, and opacity. In this alternate universe, how would stellar radii be affected for stars of the same mass as our universe?**

If the electron mass doubles, the $\boxed{\text{mean molecular weight would increase slightly}}$, because there's more mass per ionized particle—but not much because $2m_e$ is still much smaller than m_p. (The change will only be on the order of $m_e/m_p \sim 10^{-5}$.)

The electron scattering opacity is given by $\kappa_T = \sigma_T \frac{n_e}{\rho}$. Since $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$, $\kappa_T \propto m_e^{-2}$. Doubling the electron mass will therefore $\boxed{\text{decrease } \kappa_T \text{ by a factor of 4}}$.

The equation of radiative diffusion says $\mathrm{d}T/\mathrm{d}r \propto \frac{\kappa \rho L}{r^2 T^3}$. Because we're considering scaling relations, we can rewrite this using T_c and R:

$$\frac{\mathrm{d}T}{\mathrm{d}r} \propto \frac{\kappa \rho L}{R^2 T_c^3} \tag{52}$$

We can also relate $\mathrm{d}T/\mathrm{d}r$ to M using a scaling relation:

$$\frac{\mathrm{d}T}{\mathrm{d}r} \sim \frac{T_c}{R} \tag{53}$$

We want to get rid of T_c and L in these equations. Let's use the $T_c \propto M/R$ scaling (equation 35) and the scaling relation for energy generation: $\mathrm{d}L/\mathrm{d}m \sim L/M \propto \rho T_c^{17}$. Also, plug in $\rho \propto M/R^3$. This gives us

$$\frac{\mathrm{d}T}{\mathrm{d}r} \propto \frac{\kappa \rho^2 M T_c^{17}}{R^2 T_c^3} \propto \frac{\kappa M^3 T_c^{14}}{R^8} \propto \frac{\kappa M^{17}}{R^{22}} \tag{54}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} \propto \frac{M}{R^2} \tag{55}$$

Putting equations (55) and (56) together, we find $R^{20} \propto \kappa M^{16}$, or $R \propto \kappa^{1/20} M^{4/5}$. Doubling the electron mass will therefore $\boxed{\text{decrease } R \text{ by a factor of } 4^{1/20}{=}1.07}$.