Ay123 Set 1 solutions

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1. **The scale of the Sun**

   (a) **Using the angular radius of the Sun and the radiant flux received at the top of the Earth's atmosphere, calculate the effective temperature of the Sun.**
   First, relate the angular radius of the Sun θ to the physical radius R:

   $$R = a \sin\theta, \tag{1}$$

   where a is the distance from the Earth to the Sun. (Note that you can also use the small-angle approximation $\sin\theta \approx \theta$. You'll get basically the same answer.) Then, from conservation of energy (i.e., inverse square law):

   $$L = 4\pi a^2 f \tag{2}$$

   And from the Stefan-Boltzmann equation:

   $$L = 4\pi R^2 \sigma_B T_{\text{eff}}^4 \tag{3}$$

   Set equations (2) and (3) equal, plug in equation (1) for R and solve for T_{eff}:

   $$\begin{aligned} 4\pi a^2 f &= 4\pi R^2 \sigma_B T_{\text{eff}}^4 && (4)\\ a^2 f &= (a\sin\theta)^2 \sigma_B T_{\text{eff}}^4 && (5)\\ T_{\text{eff}} &= \left(\frac{f}{\sigma_B \sin^2\theta}\right)^{1/4} && (6) \end{aligned}$$

   Plugging in given values yields $\boxed{T_{\text{eff}} = 5775\text{K}}$.

   (b) **Using Venus' orbital period of 225 days and Kepler's laws, what is the semi-major axis of Venus (in AU)?**
   Start with Kepler's third law:

   $$\Omega^2 = \frac{G(M_\odot + M)}{a^3} \tag{7}$$

   First, consider the Earth-Sun system, assuming that $M_E \ll M_\odot$:

   $$\Omega_E^2 = \frac{GM_\odot}{a_E^3} \tag{8}$$

   Now, Ω_E can be calculated from known values (Earth's period is 1 year) and a_E is known (it's 1 AU), so can solve for $M_\odot$:

   $$M_\odot = \frac{\Omega_E^2 a_E^3}{G} \tag{9}$$

   Then consider the Venus-Sun system, again assuming that $M_V \ll M_\odot$:

   $$\Omega_V^2 = \frac{GM_\odot}{a_V^3} \tag{10}$$

Then solve for a_V, substituting equation (9) for $M_\odot$ and converting angular velocities to periods ($P = \frac{2\pi}{\Omega}$):

$$a_V = \left(\frac{GM_\odot}{\Omega_V^2}\right)^{1/3} \tag{11}$$
$$= \left(\frac{\Omega_E^2 a_E^3}{\Omega_V^2}\right)^{1/3} \tag{12}$$
$$= \left(\frac{P_V}{P_E}\right)^{2/3} a_E \tag{13}$$

Plug in known values ($P_E = 365.25$ days, $a_E = 1$ AU) and given values ($P_V = 225$ days), and find that $\boxed{a_V = 0.724 \text{ AU}}$.

(c) **At conjunction with the Sun, it takes astronomers on Earth 276s to detect the radio waves that reflect off Venus. Assuming circular orbits for the Earth and Venus, compute the distance between in 1 AU.**

Note: "conjunction" = Venus is directly between Earth and the Sun. Call the distance between Earth and Venus d_V. The light takes time t to travel distance $2d_V$:

$$d_V = \frac{ct}{2} = 4.14 \times 10^{12} \text{ cm} \tag{14}$$

From the previous problem, we can compute d_V in terms of AU:

$$d_V = a_E - a_V = (1 - 0.724) \text{ AU} = 0.276 \text{ AU} \tag{15}$$

Then combine equations (14) and (15) to convert AU to cm: $\boxed{1 \text{ AU} = 1.5 \times 10^{13} \text{ cm}}$.

(d) **Use your results above to compute the absolute mass, radius, and luminosity of the Sun in cgs.**

- Inverse square law: $L_\odot = 4\pi a_E^2 f$
  Plug in $a_E = 1 \text{ AU} = 1.5 \times 10^{13}$ cm and f from problem (1a).
  Find that $\boxed{L_\odot = 3.93 \times 10^{33} \text{ erg/s}}$.
- Kepler's third law: $M_\odot = \frac{\Omega_E^2 a_E^3}{G}$
  Plug in known values in cgs; find that $\boxed{M_\odot = 2.0 \times 10^{33} \text{ g}}$.
- Definition of angular size: $R_\odot = a_E \sin\theta$
  Plug in known values in cgs; find that $\boxed{R_\odot = 6.98 \times 10^{10} \text{ cm}}$.

2. **Stars are gases**

(a) **Provide a quantitative relation between temperature and density of a star that indicates when we can treat it as a gas, and show that it holds at the center of the Sun.**

To check if the center of the Sun can be treated as a gas, we can compare Coulomb energy to thermal energy. The ideal gas law is reasonable when the thermal energy ($\sim k_B T$) is larger than the Coulomb energy ($\sim (Ze^2)/r$). This occurs when

$$kT \gg Ze^2/r \tag{16}$$
$$r \gg Ze^2/kT \tag{17}$$

where r is the interparticle distance and Z is the charge of the ion ($Z = 1$ for a gas composed only of ionized hydrogen. By thinking of the number density n as r^{-3}, we can write r in terms of mass density: $r = n^{-1/3} = (\rho/\mu m_p)^{-1/3}$. Then we can rewrite equation (17):

$$\boxed{\rho \ll \mu m_p (k/e^2)^3 T^3} \tag{18}$$

For atomic hydrogen, $\mu = 0.5$. Plugging this in, our condition for treating a star as an ideal gas is (in cgs units):

$$\rho \ll 1.8 \times 10^{-16} T^3 \tag{19}$$

The sun's central temperature is $T_c \sim 10^7$ K. By equation (19), we require $\boxed{\rho_c \ll 6 \times 10^5 \text{ g cm}^{-3}}$. Since the sun's central density ρ_c is only ~ 150 g cm^{-3}, we may treat the sun as an ideal gas throughout, and need not consider Coulomb interactions between particles.

(b) **Use a scaling argument to determine the stellar mass at which the ideal gas assumption breaks down.**

Now we want to know how ρ scales with stellar mass M. (In the following derivation we only care about approximate scaling arguments, so don't worry about prefactors.) The internal energy can be approximated as $U \propto \frac{k_B T}{\mu m_p} M$ at the central temperature T. Now recall that by the virial theorem, the internal energy is approximately the gravitational energy $\Omega \propto GM^2/R$. Solve for T and find that $T \propto M/R$. We can then assume that all stars have roughly the same central temperature (which is actually a good approximation for main-sequence stars), so the central density $M \propto R$. Then $\rho \propto M/R^3 \propto M^{-2}$, so $M \propto \rho^{-1/2}$.

Plugging in values for the Sun, we find $M_{\text{lim}} = \left(\frac{\rho_{\text{lim}}}{\rho_\odot}\right)^{-1/2} M_\odot$. This yields a limiting mass of $\boxed{M = 0.016 M_\odot}$, which is about the mass of brown dwarfs or giant planets (not stars!). Therefore, we will never have to consider Coulomb interactions for main sequence stars.

3. **A toy star** Assume that a star obeys the density model

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right). \tag{20}$$

(a) **Find an expression for the central density in terms of R and M.**

Solve for total mass M by integrating over the density profile:

$$\begin{aligned} M &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^R r^2 \rho(r) \mathrm{d}r & (21)\\ &= 4\pi \int_0^R \rho_c \left(r^2 - \frac{r^3}{R}\right) \mathrm{d}r & (22)\\ &= 4\pi\rho_c \left(\frac{R^3}{3} - \frac{R^3}{4}\right) & (23)\\ &= \frac{\pi}{3}\rho_c R^3 & (24) \end{aligned}$$

Then solve for central density: $\boxed{\rho_c = \frac{3M}{\pi R^3}}$.

(b) **Find the pressure as a function of radius.**

Doing the same integral as in the previous problem, we know that

$$m(r) = \frac{4\pi}{3}\rho_c r^3 \left(1 - \frac{3r}{4R}\right) \tag{25}$$

Now use the equation of hydrostatic equilibrium:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{Gm}{r^2}\rho(r) & (26)\\ &= -G\frac{4\pi}{3}\rho_c r \left(1 - \frac{3r}{4R}\right) \rho_c \left(1 - \frac{r}{R}\right) & (27)\\ &= -\frac{4\pi}{3} G \rho_c^2 r \left(1 - \frac{7r}{4R} + \frac{3r^2}{4R^2}\right) & (28) \end{aligned}$$

Integrate equation (28) to get the total pressure:

$$P(r) = -\frac{4\pi}{3}G\rho_c^2 \int \left(r - \frac{7r^2}{4R} + \frac{3r^3}{4R^2}\right) \mathrm{d}r \tag{29}$$

$$= -\frac{4\pi}{3}G\rho_c^2 \left[\frac{r^2}{2} - \frac{7r^3}{12R} + \frac{3r^4}{16R^2} + C\right] \tag{30}$$

Use the zero boundary condition ($P(R) = 0$) to solve for the integration constant C:

$$P(R) = -\frac{4\pi}{3}G\rho_c^2 R^2 \left[\frac{1}{2} - \frac{7}{12} + \frac{3}{16} + \frac{C}{R^2}\right] = 0 \tag{31}$$

$$C = R^2\left(-\frac{1}{2} + \frac{7}{12} - \frac{3}{16}\right) \tag{32}$$

$$= -\frac{5}{48}R^2 \tag{33}$$

Now let's solve for the central pressure $P_c = P(r = 0)$:

$$P_c = -\frac{4\pi}{3}G\rho_c^2 C \tag{34}$$

$$= \frac{5\pi}{36}G\rho_c^2 R^2 \tag{35}$$

Okay, finally we can substitute stuff into equation (30) to write the full equation for pressure:

$$P(r) = -\frac{4\pi}{3}G\rho_c^2 \left[\frac{r^2}{2} - \frac{7r^3}{12R} + \frac{3r^4}{16R^2} - \frac{5}{48}R^2\right] \tag{36}$$

$$= P_c\left[1 - \frac{24}{5}\left(\frac{r}{R}\right)^2 + \frac{28}{5}\left(\frac{r}{R}\right)^3 - \frac{9}{5}\left(\frac{r}{R}\right)^4\right] \tag{37}$$

So we find that $\boxed{P(r) = P_c \times f\left(\frac{r}{R}\right)}$ as expected.

Now plug in the answer for part (a) to get P_c as a function of M and R:

$$P_c = \frac{5\pi}{36}G\left(\frac{3M}{\pi R^3}\right)^2 R^2 \tag{38}$$

We can simplify this to $\boxed{P_c = \frac{5}{4\pi}\frac{GM^2}{R^4}}$.

(c) **What is the central temperature T_c, assuming an ideal gas equation of state? How does it scale with mean particle mass?**

Ideal gas: $P = \frac{\rho k_B T}{\mu m_p}$

Solve this for central temperature, plugging in answer from (b) for P_c and answer from (a) for ρ_c:

$$T_c = \frac{P_c \mu m_p}{\rho_c k_B} \tag{39}$$

$$= \frac{\frac{5\pi}{36}G\rho_c R^2 \mu m_p}{k_B} \tag{40}$$

$$= \frac{\frac{5\pi}{36}G\frac{3M}{\pi R^3}R^2 \mu m_p}{k_B} \tag{41}$$

This simplifies to $\boxed{T_c = \frac{5GM}{12R}\frac{\mu m_p}{k_B}}$ which scales as $\boxed{T_c \propto \mu m_p}$. ($\mu m_p$ is the mean particle mass.)

(d) **Find the ratio of radiation pressure to gas pressure at the center of the star as a function of total stellar mass in $M_\odot$. At what mass does radiation pressure become comparable to the ideal gas pressure?**

Radiation pressure is given by $P_{\text{rad}} = \frac{1}{3}a_o T^4$. The ratio at the center of the star is therefore

$$\frac{P_{\text{rad}}}{P_{\text{gas}}} = \frac{1}{3}a_o \frac{T_c^4}{\frac{5}{4\pi}\frac{GM^2}{R^4}} \tag{42}$$

Then plug in T_c from part (c):

$$\begin{aligned} \frac{P_{\text{rad}}}{P_{\text{gas}}} &= \frac{1}{3}a_o \frac{\left(\frac{5GM}{12R}\frac{\mu m_p}{k_B}\right)^4}{\frac{5}{4\pi}\frac{GM^2}{R^4}} \quad (43) \\ &= \frac{125\pi}{15552}a_o G^3 M^2 \left(\frac{\mu m_p}{k_B}\right)^4 \quad (44) \end{aligned}$$

Assuming solar composition ($\mu = 0.62$), we can rewrite this in terms of solar masses as:

$\boxed{\frac{P_{\text{rad}}}{P_{\text{gas}}} = 7.2 \times 10^{-4}\left(\frac{M}{M_\odot}\right)^2}$ When $\frac{P_{\text{rad}}}{P_{\text{gas}}} = 1$, the mass of the star is $\boxed{M \approx 37 M_\odot}$. Note that this is not an exact result, since our formula for T_c assumes that radiation pressure is negligible.

(e) **Write the total gravitational potential energy of this toy star and verify the virial theorem.**

The total gravitational potential is $\Omega = -\int \frac{Gm(r)}{r}\mathrm{d}m$. For simplicity's sake, let's convert this to an integral over radius so we can plug in the definition of $m(r)$ from part (b) and the definition of $\rho(r)$:

$$\begin{aligned} \Omega &= -\int_0^R Gm(r)4\pi r\rho \mathrm{d}r \quad (45) \\ &= -\int_0^R G\frac{4\pi}{3}\rho_c r^3\left(1 - \frac{3r}{4R}\right)4\pi r\rho_c\left(1 - \frac{r}{R}\right)\mathrm{d}r \quad (46) \\ &= -G\frac{16\pi^2}{3}\rho_c^2\int_0^R\left(r^4 - \frac{7r^5}{4R} + \frac{3r^6}{4R^2}\right)\mathrm{d}r \quad (47) \\ &= -G\frac{16\pi^2}{3}\rho_c^2\left[\frac{R^5}{5} - \frac{7R^6}{24R} + \frac{3R^7}{28R^2}\right] \quad (48) \end{aligned}$$

The gravitational potential is $\boxed{-G\frac{26\pi^2}{315}\rho_c^2 R^5}$

Now let's do math with the other side of the virial theorem, plugging in $P(r)$ from equation (34):

$$\begin{aligned} -3\int P\mathrm{d}V &= -3\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta \mathrm{d}\theta\int_0^R r^2 P(r)\mathrm{d}r \quad (49) \\ &= 12\pi\int_0^R r^2\frac{4\pi}{3}G\rho_c^2\left(\frac{r^2}{2} - \frac{7r^3}{12R} + \frac{3r^4}{16R^2} - \frac{5}{48}R^2\right)\mathrm{d}r \quad (50) \\ &= \frac{48\pi^2}{3}G\rho_c^2\int_0^R\left(\frac{r^4}{2} - \frac{7r^5}{12R} + \frac{3r^6}{16R^2} - \frac{5r^2}{48}R^2\right)\mathrm{d}r \quad (51) \\ &= \frac{48\pi^2}{3}G\rho_c^2\left[\frac{R^5}{10} - \frac{7R^5}{72} + \frac{3R^5}{112} - \frac{5R^5}{144}\right] \quad (52) \\ &= -\frac{26\pi^2}{315}G\rho_c^2 R^5 \quad (53) \end{aligned}$$

So $\boxed{-3\int P\mathrm{d}V = \Omega}$, and the virial theorem is exactly satisfied.

4. **Stellar coronae**

(a) **Solve hydrostatic equilibrium for the density profile as a function of radius given a density ρ_b at the base radius $r_b \approx R$.**

Start with hydrostatic equilibrium and assume an ideal gas equation of state with constant temperature T. Also assume that the mass of the corona is negligible compared to the mass of the star M, so $m(r) \approx M$:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)}{r^2}\rho(r) \tag{54}$$

$$\frac{k_B T}{\mu m_p}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM}{r^2}\rho(r) \tag{55}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM\mu m_p}{r^2 k_B T}\rho(r) \tag{56}$$

Equation (56) is a separable differential equation:

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{GM\mu m_p}{k_B T}\frac{\mathrm{d}r}{r^2} \tag{57}$$

$$\rho(r) = A\exp\left(\frac{GM\mu m_p}{r k_B T}\right) \tag{58}$$

Now use the given boundary condition to solve for integration constant A:

$$\rho_b = A\exp\left(\frac{GM\mu m_p}{r_b k_B T}\right) \tag{59}$$

$$A = \rho_b\exp\left(-\frac{GM\mu m_p}{r_b k_B T}\right) \tag{60}$$

So the final equation is

$$\boxed{\rho(r) = \rho_b\exp\left(-\frac{GM\mu m_p}{r_b k_B T}\right)\exp\left(\frac{GM\mu m_p}{r k_B T}\right)} \tag{61}$$

(b) **What is the pressure in the corona as $r \to \infty$? Comment on the implications of a low-density, low-pressure ISM surrounding the star.**

As $r \to \infty$, we find the limit $\rho(r) \to \rho_b\exp\left(-\frac{GM\mu m_p}{r_b k_B T}\right)$. Since we've assumed an ideal gas ($P = \frac{\rho k_B T}{\mu m_p}$, this yields a pressure $\boxed{P \to \frac{\rho_b k_B T}{\mu m_p}\exp\left(-\frac{GM\mu m_p}{r_b k_B T}\right)}$. This is finite!

Since the stellar corona is surrounded by an ISM with much lower pressure, we expect the corona to be able to produce a stellar wind.

5. **Using the MESA stellar evolution code**

(a) **Run the default stellar model located in mesa/star/work/. At time step 10, what is the core temperature and surface temperature of the model?**

The core temperature is $\log T_c = 5.48$ [K], or $\boxed{T_c = 3.012 \times 10^5\ \mathrm{K}}$. The surface temperature is $\boxed{T_{\mathrm{eff}} = 3452\ \mathrm{K}}$.

(b) **Evolve a $1M_\odot$ model up the red giant branch. Make an HR diagram and plot $\log L$ as a function of time.**

Recall that an H-R diagram should have temperature increasing from right to left. Also note that in astronomy, log means $\log_{10}$ and *not* ln. Finally, *make sure to include units!*