Ay123 Problem Set 5

Due Wednesday, December 5, 9:00 am

1. **Supernova Shock Revival from Neutrino Heating** (8 points)

   Consider the final iron core of a massive star with a mass $M_{\rm Fe} = 1.5\ M_\odot$ and radius $R_{\rm Fe} = 3 \times 10^8$ cm. When this core collapses, the initial collapse stops when the central core with a mass $M_{\rm core} = 0.7\ M_\odot$ reaches nuclear densities. At this density the core bounces, driving a shock with an energy $E_{\rm bounce} = 10^{51}$ erg into the infalling outer core.

   (a) Estimate the energy that is required to photodissociate 0.8 $M_\odot$ of Fe into alpha particles and neutrons. Compare this energy to the bounce shock energy and comment on the fate of the shock.

   (b) In the proto-neutron star (with an initial radius 2×10^6 cm), the mean free path of neutrinos is $l_\nu = 30$ cm. Estimate the diffusion time for neutrinos to escape from the proto-neutron star and hence estimate the neutrino luminosity during the initial neutron-star cooling phase.

   (c) Assuming that 10% of the neutrino luminosity is absorbed by the infalling outer core, estimate how long it takes to absorb enough neutrino energy to reverse the infall of the $0.8\,M_\odot$ outer core and drive a successful supernova explosion with a typical explosion energy of 10^{51} erg. Assume the outer core has initial energy per unit mass $\epsilon = -GM_{\rm Fe}/R_{\rm Fe}$. Compare this time to the dynamical (free-fall) timescale of the proto-neutron star.

2. **Protostar** (10 points)

   (a) Find the average density and central temperature (as a function of mass) of an accreting protostar whose initial radius is given by the expression

   $$\frac{R}{R_\odot} = \frac{43.2}{1 - 0.2X}\frac{M}{M_\odot}$$

   if its structure is approximated by a $n = 1.5$ polytrope with hydrogen mass fraction $X = 0.7$ and helium fraction $Y = 0.3$.

   (b) Suppose the protostar maintains a polytropic structure until its collapse is halted when the central temperature reaches $T_{\rm crit}$ required for hydrogen burning. Show that the greater the mass of the star, the smaller the density at the point where $T_{\rm crit}$ is reached:

   $$\rho_{\rm crit} = 1.52\,\frac{1}{M^2}\left(\frac{k_B T_{\rm crit}}{\mu m_{\rm H} G}\right)^3$$

   (c) Noting the criterion for electron degeneracy, estimate the critical mass below which collapse is halted by electron degeneracy, not by hydrogen burning. After dropping factors of order unity, show that this mass is related to the Chandrasekhar limit, $M_{\rm Ch}$, by the approximate relation

   $$\frac{M_{\rm crit}}{M_{\rm Ch}} \sim \left(\frac{\mu_e}{\mu}\right)^{3/2}\left(\frac{k_B T_{\rm crit}}{m_e c^2}\right)^{3/4}$$

   Evaluate this mass for $T_{\rm crit} = 5 \times 10^6$ K and $M_{\rm Ch} = 1.4 M_\odot$.

3. **Binary Stars** (10 points)

   The minimum orbital separation of a star with mass M and radius R in a binary star system is

   $$a_{\rm min} \simeq \frac{5}{2}\left(\frac{M_{\rm tot}}{M}\right)^{1/3} R\,,$$

   where $M_{\rm tot}$ is the total mass of the binary system.

(a) Show that the minimum orbital period of the binary is

$$P_{\rm min} \simeq 5\pi \left(\frac{15}{8\pi}\right)^{1/2} (G\rho)^{-1/2} ,$$

where ρ is the average stellar density. Evaluate $P_{\rm min}$ for a binary system of two red giants with $\rho = 10^{-6}\,{\rm g/cm^3}$, two Sun-like stars with $\rho = 1\,{\rm g/cm^3}$, two white dwarfs with $\rho = 10^6\,{\rm g/cm^3}$, and two neutron stars with $\rho = 3\times 10^{14}\,{\rm g/cm^3}$.

(b) Consider a red giant of $M_1 = 1\,M_\odot$, with a core mass $M_c = 0.5\,M_\odot$, envelope mass $M_e = 0.5\,M_\odot$, and radius $R_1 = 100\,R_\odot$. It undergoes a common-envelope event with a low-mass secondary star of mass M_2 and radius R_2, which ejects the envelope of the red giant. The α prescription for common-envelope events predicts the final orbital separation a_f:

$$\alpha\left(\frac{GM_cM_2}{2a_f} - \frac{GM_1M_2}{2a_i}\right) = \frac{GM_cM_e}{R_1} \,. \tag{1}$$

Solve equation 1 for a_f. Show that when α is of order unity and $M_2 \ll M_e$, the final orbital separation satisfies $a_f \ll a_i$, and equation 1 reduces to

$$a_f \simeq \frac{\alpha}{2}\frac{M_2}{M_e} R_1 \,.$$

(c) A stellar merger will occur if the final separation a_f between the secondary and the primary's core is smaller than the minimum orbital separation possible for the secondary star. By replacing a_f with $a_{\rm min}$ for the secondary, and using $M_2 \ll M_c$, find the minimum secondary mass that can eject the envelope of the primary without merging with the core of the primary. Evaluate this mass for $\alpha = 0.5$ and typical brown dwarf radius $R_2 = 0.1R_\odot$.

4. **Hydrogen Lines from Stars** (12 points)

(a) Consider a stellar atmosphere of pure hydrogen gas. Let's suppose H atoms only populate the $n = 1$ (ground) and $n = 2$ states. If n_2 is the number density of atoms with electrons in the $n = 2$ state, write down an expression for $n_2/n_{\rm tot}$. You will need to use the Boltzmann factor in addition to your result from the Saha equation Problem 3a of HW 3.

(b) If the continuum photosphere is at a total number density $n_{\rm tot} = 10^{17}{\rm cm}^{-3}$, make a plot of $n_2/n_{\rm tot}$ as a function of stellar surface temperature. Recall that the energy levels of the H atom are given by $E = -13.6/n^2$ eV and the degeneracies are $g_n = 2n^2$. At what temperature does the value of $n_2/n_{\rm tot}$ peak? If the strength of Balmer lines is determined by the relative population $n_2/n_{\rm tot}$, which stellar spectral type should show the most prominent H lines? Although you will not be very far off, you should get the *wrong* answer in this part of the problem. In reality, A-type stars with $T_{\rm eff} = 10000$ K have the strongest Balmer lines.

(c) The cross-section at line center for the production of Balmer lines is $\sigma \simeq 10^{-16}$ cm^2. Assuming an isothermal atmosphere for an A-type star with $g = 10^4{\rm cm\,s^{-2}}$, calculate the star's scale height. Then assume a value $n_2/n_{\rm tot} = 10^{-4}$ that is constant, and compute the optical depth at the center of the Balmer line at the continuum photosphere of an A-type star. Is this small or large? Calculate the number density at which $\tau = 1$ near the center of the Balmer line. At how many scale heights above the continuum photosphere of the star is the Balmer line formed?

(d) Replot the value of $n_2/n_{\rm tot}$ as in part b, but with the number density you computed in part c. At which temperature do you now expect Balmer lines to be strongest?

5. **MESA Project, Part 2** (10 points)

Complete the MaxiLab associated with the MiniLab you completed on Homework #4. Hand in answers to questions and plots created during the course of the MaxiLab.