Ay123 Problem Set 4

Due Wednesday, November 21, 9:00 am

1. **Cepheid Variable** (8 points)

   The mass and mean radius of a typical Cepheid variable are $\log(M/M_\odot) = 0.8$ and $\log(R/R_\odot) = 1.4$.

   (a) Use the continuity equation to show that a radial perturbation that satisfies $\Delta\rho/\rho = -\Delta V/V$ (where V is volume) implies that
   $$\frac{\partial}{\partial r}\frac{\Delta r}{r} = 0\,, \tag{1}$$
   where Δr is the radial Langrangian displacement.

   (b) For a radial pulsation satisfying equation 1, use the continuity equation to relate $\Delta\rho/\rho$ to $\Delta r/r$.

   (c) Use this relation in the momentum equation to show that $\omega^2 = (3\gamma - 4)g/r$. What does this imply about the stability of the star when $\gamma < 4/3$?

   (d) Using $\gamma = 5/3$, for a pulsation amplitude $\Delta r/r_0 = 0.1$, calculate the maximum surface velocity of the star in cgs units.

   (e) Compute the fractional surface temperature perturbation $\Delta T_{\rm eff}/T_{\rm eff}$ and luminosity perturbation $\Delta L/L$.

2. **Helioseismology** (10 points)

   Assume the Sun is fully ionized and fully convective.

   (a) Show that the sound speed close the surface is given by $c_s^2 = (\gamma - 1)gz$, where $z = R - r \ll R$ is the distance from the surface. You may assume ρ, T, and P are zero at the surface.

   (b) For a horizontal wave number, $k_\perp$, and wave frequency, ω, find the maximum distance $z_{\rm max}$ that an acoustic wave can penetrate into the star. Write this result in terms of the associated spherical harmonic ℓ. Do acoustic waves of higher frequency penetrate deeper or shallower into the star? What about waves of higher ℓ?

   (c) Assume an oscillation mode has a radial displacement that is zero at $z = 0$ and $z = z_{\rm max}$. Show that
   $$\int_0^{z_{\rm max}} k_r\,dz = (n+1)\pi \tag{2}$$
   where k_r is the radial wavenumber and n is the number of nodes in the radial wavefunction.

   (d) From equation 2, use the acoustic wave dispersion relation to find a relation between n, l, and ω.

   (e) For $l = 0$, the appropriate value of $z_{\rm max}$ is R. Evaluate equation 1 for $l = 0$ using the acoustic dispersion relation, still assuming g is constant (which is a reasonable approximation because the integral is dominated by the near-surface regions of the star where g is nearly constant). Show that the frequency spacing between modes of successive n is proportional to the square root of the density of the star.

3. **Nuclear Cross Sections** (10 points)

You may wish to read KWW §18.3 for this problem. The cross section for a nuclear reaction is a result of quantum tunneling. It is a function of v (or equivalently, E) evaluated in the center of mass of the reacting particles A and B. In an equilibrium gas, it can be averaged over a Maxwellian velocity distribution:

$$\langle \sigma v \rangle = 4\pi \left(\frac{m_{AB}}{2\pi k_{\rm B} T} \right)^{3/2} \int_0^\infty v \frac{S(E)}{E} \exp\left(-\frac{m_{AB} v^2}{2 k_{\rm B} T} \right) \exp\left(-\frac{2\pi Z_A Z_B e^2}{\hbar v} \right) v^2 \, dv \, ,$$

where m_{AB} is the reduced mass of A and B.

(a) To simplify this integral a bit, write it in terms of E and b, where $E = m_{AB} v^2/2$ and

$$b = \frac{\sqrt{2 m_{AB}}\, \pi Z_A Z_B e^2}{\hbar} = 0.99\, Z_A Z_B \sqrt{m_{AB}}\ (\mathrm{MeV})^{1/2} \, .$$

(b) Recast the integral as the integral of a Gaussian. The Gaussian is centered at E_0 with a width Δ and an amplitude C:

$$g(E) = C \exp\left(-\frac{(E - E_0)^2}{(\Delta/2)^2} \right)$$

Find the values of C, E_0, and Δ, and do the integral.

(c) Derive the expression

$$\langle \sigma v \rangle \propto \frac{1}{Z_A Z_B m_{AB}} S_0 \tau^2 e^{-\tau}$$

and give the expression for τ.

(d) The nuclear energy generation rate scales with temperature as $\epsilon = \epsilon_0 \rho T^\eta$. Find η for the reactions

i. $p + p \to d + e^+ + \nu_e$ at $T = 1.0 \times 10^7$ K and 1.5×10^7 K

ii. $^7\mathrm{Be} + p \to {}^8\mathrm{B} + \gamma$ at $T = 1.5 \times 10^7$ K

iii. $^{14}\mathrm{N} + p \to {}^{15}\mathrm{O} + \gamma$ at $T = 1.5 \times 10^7$ K and 2.5×10^7 K

4. **Deuterium Burning** (12 points)

Small amounts of Deuterium are made in the Big Bang. D is destroyed in the interiors of stars via the reaction $p + D \to {}^3He + \gamma$. The value of S_0 for D-burning is 2.5×10^{-4} keV-barn, each reaction releases ≈ 5.5 MeV, and the cosmic abundance of D from the Big Bang is $n_D \approx 2 \times 10^{-5} n_H$. Let's focus on a low mass fully convective star undergoing KH contraction; such a star can be reasonably well modeled as an $n = 3/2$ polytrope.

(a) What are the values of m_{AB}, Z_A, Z_B, and b^2 (defined in problem 1) for Deuterium burning? Write down the resulting thermally averaged value of $\langle \sigma v \rangle$ for D fusion.

(b) The luminosity of a protostar on the Hayashi track is

$$L \simeq 0.2\, L_\odot \left(\frac{M}{M_\odot} \right)^{1/2} \left(\frac{R}{R_\odot} \right)^2 . \qquad (3)$$

By equating this with the energy released by Kelvin-Helmholtz contraction, calculate the local contraction time t_c as a function of the mass and radius of the star. Does the contraction time get shorter or longer as the star contracts?

(c) What is the lifetime t_D of a D nucleus at the center of the star in terms of the local density and temperature? Use the properties of $n = 3/2$ polytropes to write t_D as a function of M and R. Does the D lifetime get shorter or longer as the star contracts?

(d) For any mass M show that there is a critical radius R_D at which $t_D = t_c$. This represents the radius (time) at which D starts to undergo significant fusion. Give the numerical value of R_D for $M = 0.03$ and $0.1\,M_\odot$. For each of these two cases, also determine the central temperature of the star T_c and the D lifetime t_D when $R = R_D$. Does D fusion occur before or after the star reaches the main sequence?

(e) Explain quantitatively whether D fusion can halt (at least temporarily) the KH contraction of the star. If so, how long does the "D main sequence" last for the two cases considered in part d) above?

5. **MESA Project, Part 1** (10 points)

Look through the list of recent MESA summer school labs at

http://cococubed.asu.edu/mesa_market/summer_school_material.html

and

http://cococubed.asu.edu/mesa_summer_school_2018/agenda.html

(a) Choose one of the labs from days Tuesday-Friday, i.e., lab assignments that contain two minilabs and one maxilab. Options include labs by Charlie Conroy, Jenn van Saders, Warrick Ball, Matteo Cantiello, Jonathan Fortney, Selma de Mink, Pablo Marchant, Leslie Rogers, Norbert Langer, Alfred Gautschy, Conny Aerts, Jim Fuller (2016), Ken Shen, Lars Bildsten (2015), Craig Wheeler, Sterl Phinney, Pascale Garaud, Eliot Quataert, and Dean Townsley. Labs before 2014 are discouraged as MESA has evolved a lot since then. You may want to look at the introductory lab, Beyond Inlists by Josiah Schwab, to become more familiar with MESA.

(b) Download the labs and associated lecture, and follow the instructions within the labs/lectures to carry out **both MiniLabs**. Write answers and generate plots to answer any questions asked within the labs. You will be doing the MaxiLab in your next homework.