# Ay123 Problem Set 2

due Wednesday, October 24, 9:00 am

1. **Convection** (10 points)

   In class, we showed that the buoyant acceleration felt by a fluid element displaced radially by $\Delta r$ is

$$\begin{aligned} a &= g\left[\frac{d\ln\rho}{dr} + \frac{g}{c_s^2}\right]\Delta r \\ &= -N^2\Delta r \end{aligned} \tag{1}$$

   where $N$ is the Brunt-Vaisala frequency and $c_s^2 = \gamma P/\rho$. You may assume the unperturbed star is spherical and in hydrostatic equilibrium.

   (a) The Brunt-Vaisala frequency can also be expressed

$$N^2 = \frac{g}{H}\left(\nabla_{\rm ad} - \nabla\right),$$

   where $H = P/(\rho g)$ is the pressure scale height. The mixing length theory of convection envisions blobs of material that advect energy after traveling a "mixing length" $\Lambda$, parameterized in terms of the pressure scale height as $\Lambda = \alpha H$. If a convective blob accelerates from zero velocity over a mixing length according to

$$\frac{d}{dr}v_{\rm con} = |N|,$$

   find the maximum convective velocity $v_{\rm con}$ in terms of $N$ and the mixing length $\Lambda$. You may assume $N$ is constant over a mixing length. Express $v_{\rm con}$ in terms of $\alpha$, $\gamma$, $(\nabla_{\rm ad} - \nabla)$, and $c_s$.

   (b) Using your expression for $v_{\rm con}$ from above, express the kinetic energy flux $F_{\rm con} = \rho v_{\rm con}^3$ of upgoing convective blobs.

   (c) The base of the Sun's convection zone has radius $r \sim 5\times10^{10}$ cm, density $\rho \sim 10^{-1}$ g/cm$^3$, sound speed $c_s \sim 2\times10^7$ cm/s, and scale height $H \sim 5\times10^9$ cm. For $\alpha = 2$, what is the value of $(\nabla_{\rm ad} - \nabla)$ required for convection to carry the Sun's luminosity? What is the corresponding maximum convective velocity $v_{\rm con}$, and how does this compare to the sound speed $c_s$?

   (d) Assuming convection carries all the Sun's luminosity, use the expression for $N^2$ in equation 1 to show that the density gradient in a convection zone is given by

$$\frac{d\ln\rho}{dr} = \frac{g}{c_s^2}\left[\frac{\gamma^2}{\alpha^2}\left(\frac{L_\odot}{4\pi\rho r^2 c_s^3}\right)^{2/3} - 1\right] \tag{2}$$

   Using the Sun's properties from part e, determine which of the terms in the brackets in equation 2 is larger. Will the Sun's density profile be strongly dependent on $\alpha$?

   (e) The Sun's surface has radius $r \sim 7\times10^{10}$ cm, density $\rho \sim 10^{-7}$ g/cm$^3$, sound speed $c_s \sim 8\times10^5$ cm/s, and scale height $H \sim 10^7$ cm. For $\alpha = 2$, what is value of $(\nabla_{\rm ad} - \nabla)$ required for convection to carry the Sun's luminosity? What is the corresponding convective velocity, and how does it compare to the sound speed? Will the value of $d\ln\rho/dr$ depend on $\alpha$ near the surface of the Sun?

2. **Fully convective cool stars** (15 points)

   Imagine a star that is sufficiently centrally concentrated that the mass of the envelope is negligible to the mass in the core, $M$. Assume that the envelope is convective and composed of an ideal gas with polytropic index $\gamma$.

   (a) Show that the temperature within the envelope is

   $$T = \frac{\gamma}{\gamma - 1} \frac{GM\mu m_H}{k_B} \left( \frac{1}{r} - \frac{1}{R} \right) + T_s$$

   where $T_s$ is the surface temperature.

   (b) In cool stars, the polytropic convective envelope extends nearly all the way to the photosphere. Use this fact to derive a scaling between pressure and temperature at the photoshpere.

   (c) Assuming nearly constant opacity above the photosphere, show that the photospheric pressure is

   $$P_s = \frac{2g_s}{3\kappa_s}$$

   where $\kappa_s$ is the photospheric opacity.

   (d) In stars cooler than the Sun, H- opacity is the dominant opacity near the photosphere, and has approximate dependence $\kappa_{H-} \propto T^9$. Use this fact, and your results from parts b and c, to determine a scaling between $M, R, T_{\text{eff}}$ for cool stars. Also express this scaling in terms of $M, R, L$, and $M, T_{\text{eff}}, L$. **Hint:** You'll need to include the mass/radius dependence of the polytropic constant $K$, see HKT Chapter 7.

   (e) Main sequence G/K/M stars approximately have $L \propto M^3$. Draw the lower part of the main sequence on an HR dagram, from $0.1\,M_\odot < M < 1\,M_\odot$.

   (f) As stars evolve up the red giant branch, they become nearly fully convective, their mass remains nearly constant, and their luminosity increases. Derive the scaling between luminosity and temperature for a red giant, and add an evolutionary track for a red giant branch star on an HR diagram.

3. **Hot Massive Stars** (10 points)

   Consider a family of stars in which the opacity is dominated by Thomson (electron) scattering and in which nuclear energy is generated by the CNO cycle.

   (a) How do opacity and energy generation depend on density and temperature?

   (b) In analogy with the homology relations that we derived in Lecture V, find the relation between radius and mass. Also find the relation between luminosity and mass.

   (c) Where is this family of stars on the Hertzsprung–Russell diagram (luminosity vs. temperature)?

   (d) Generate models of $10\,M_\odot$, $20\,M_\odot$, $30\,M_\odot$, $40\,M_\odot$, and $60\,M_\odot$ models with MESA. Record their radius, luminosity, and temperature early on the main sequence (when the central hydrogen abundance is about 60%). Compare these results to your homology relations above.

   (e) Consider a universe where electrons are twice as massive. How would this affect the mean molecular weight of stars? How would this affect the electron scattering opacity? From the equation of radiative diffusion, derive a scaling relation between stellar radius, mass, and opacity (Hint: use result from above that $T_c \propto M/R$, and $\rho \propto M/R^3$). In this alternate universe, how would stellar radii be affected for stars of the same mass as our universe?

4. **Realistic stellar structures with MESA** (15 points)

   (a) Output a stellar profile when the age of your model is approximately 4.6 Gyr, the current age of the Sun.

(b) Using your `profile*.data` file, make a plot of $\log \rho$, $\log P$, $\log T$ vs. radius for your model. By computing numerical derivatives, also compute and plot $d \log P / d \log \rho$ as a function of radius. Indicate (e.g., by using a thick or colored line) which parts of the star are convective. Where and why is $d \log P / d \log \rho$ nearly constant?

(c) Using your own integrator (written in any language you choose), numerically integrate the Lane-Emden equations for a polytrope of $n = 3/2$ and $n = 3$. Add these lines to your plots of $\log P$ and $\log \rho$ above, normalized to have the same central pressure and density, and same radius as the Sun. Which polytrope better approximates the internal structure of the Sun?

(d) Plot the nuclear energy generation rate, `eps_nuc`, as a function of radius. Make sure you edit your `profile_columns.list` file so that this quantity is included in the output. Compare `eps_nuc` to the approximate pp-energy generation rate $\epsilon = \epsilon_0 \rho T^4$, and estimate the proportionality constant $\epsilon_0$.

(e) Evolve your star until it reaches a luminosity of $L \simeq 100\, L_\odot$, and repeat item (b). Are red giants more or less centrally condensed than main sequence stars? Are their surface convective zones thicker or thinner?