

# Ay 123 Lecture VI, Part b Excitation and Ionization

An ideal gas equation of state can be derived from the same distribution function. It occurs in the limit

$$\frac{-\mu}{k_B T} \gg 1$$

This occurs, for instance, in an ionized gas where  $\mu$  is very large because only pair creation could create a particle, and so we would have  $\mu \sim mc^2 \gg k_B T$ . It could also occur for a very cold atomic gas where the temperature is small, such that  $k_B T \ll \mu_{\text{ion}}$ , where  $\mu_{\text{ion}}$  is the ionization potential for the atoms in the gas. The first scenario is common in stellar interiors, the second in planetary atmospheres.

When  $\frac{-\mu}{k_B T} \gg 1$ , the distribution function simplifies

$$\frac{dN}{d^3x d^3p} = \frac{1}{h^3} \sum_j \frac{g_j}{\underbrace{e^{-\mu/k_B T}}_{\gg 1} e^{E_j/k_B T} e^{E(p)/k_B T} + 1}$$

$$\approx \frac{1}{h^3} \sum_j g_j e^{\mu/k_B T} e^{-E_j/k_B T} e^{-E(p)/k_B T}$$

We'll consider non-relativistic particles, so  $E(p) = \frac{p^2}{2m}$ ,  $v = \frac{p}{m}$ . The number density is, considering for simplicity one energy state  $j$ ,

$$n = \int d^3p \frac{dN}{d^3x d^3p} = \frac{4\pi}{h^3} g_0 \int_0^\infty p^2 e^{\mu/k_B T} e^{-E_0/k_B T} e^{-\frac{p^2}{2mk_B T}} dp$$

$$\Rightarrow n = \frac{4\pi}{h^3} g_0 e^{\mu/k_B T} e^{-\epsilon_0/k_B T} (2mk_B T)^{3/2} \underbrace{\int_0^\infty x^2 e^{-x^2} dx}_{\sqrt{\pi}/4}$$

$$\Rightarrow e^{\mu/k_B T} = \frac{nh^3}{g_0 (2\pi mk_B T)^{3/2}} e^{\epsilon_0/k_B T}$$

This tells us the chemical potential. For free particles (no energy states  $\epsilon_0$ ) this gives

$$e^{\mu/k_B T} = \frac{nh^3}{(2\pi mk_B T)^{3/2}}$$

which is appropriate for an ideal gas. For typical conditions in main sequence stars, the right hand side is tiny, requiring  $\mu/k_B T \ll -1$ , validating our derivation.

Let's now consider a system with multiple available energy states  $i$ . Then, we can divide our expression for number density to get the relative population of the two states

$$\boxed{\frac{n_i}{n_0} = \frac{g_i}{g_0} e^{-\frac{(\epsilon_0 - \epsilon_i)}{k_B T}}}$$

Boltzmann Equation

This could tell us, eg, the relative population of electrons in an atom with different energy levels  $\epsilon_i$ .

Real electrons have many energy levels, so relative populations requires summation,

$$n = \sum_{i=0}^{\infty} n_i$$

$$\Rightarrow g_0 \frac{n}{n_0} = g_0 \sum_i \frac{n_i}{n_0}$$

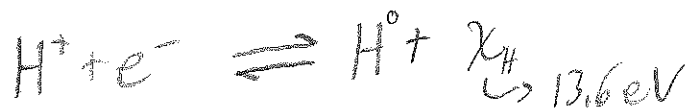
$$= g_0 + g_1 e^{-\epsilon_1/k_B T} + g_2 e^{-\epsilon_2/k_B T} + \dots$$

$\equiv U_p(T)$  Partition Function

So

$$\frac{n_i}{n} = \frac{g_i}{U_p} e^{-\epsilon_i/k_B T}$$

This tells us relative population of energy levels. What about fraction of ionized atoms?



Then

$$n_e = \frac{2(2\pi m_e k_B T)^{3/2}}{h^3} e^{\mu_e/k_B T}$$

$$n_{H^+} = \frac{(2\pi m_p k_B T)^{3/2}}{h^3} e^{\mu_{H^+}/k_B T}$$

$$n_{H^0} = \frac{2(2\pi(m_e+m_p)k_B T)^{3/2}}{h^3} e^{\mu_{H^0}/k_B T} e^{\chi_H/k_B T}$$

Now, in equilibrium,  $\mu_e + \mu_{H^+} = \mu_{H^0}$ , so we can divide:

$$\frac{n_{H^+} n_e}{n_{H^0}} = \left( \frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_H/k_B T}$$

Saha Equation

The total # density of H is  $n_H = n_{H^0} + n_{H^+}$

Degree of ionization:

$$y = \frac{n_{H^+}}{n} = \frac{n_e}{n}$$

Assuming only H composition

$$\Rightarrow \frac{y^2}{1-y} = \frac{1}{n_H} \left( \frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_H/k_B T}$$

$y$  transitions rapidly from 0 to 1 when  $\chi_H \sim k_B T$ , with some dependence on # density. Transition temperatures are

$$\begin{aligned} H \rightarrow H^+ & : T \sim 10^4 \text{ K} & \chi = 13.6 \text{ eV} \\ He \rightarrow He^+ & : T \sim 3 \times 10^4 \text{ K} & \chi = 24.6 \text{ eV} \\ He^+ \rightarrow He^{2+} & : T \sim 6 \times 10^4 \text{ K} & \chi = 54.4 \text{ eV} \end{aligned}$$

Note that a partially ionized gas changes the particle # density and therefore the equation of state

$$\mu = (1+y)^{-1} \quad \text{for pure hydrogen}$$

$$\Rightarrow p = n_H (1+y) k_B T$$

But  $y = y(P, T)$ , so EOS is complex, and is not simple ideal gas anymore. The gas internal energy is now

$$E = \underbrace{\frac{3}{2} n k_B T (1+y)}_{\text{Kinetic energy of particles}} + \underbrace{y n \chi_H}_{\text{Potential energy from recombination}}$$

### Pressure Ionization

Above calculation assumed  $e^-$  energy levels are not affected by environment. If atoms are crowded too close bound electrons of different atoms interfere. Collisions between atoms can "knock"  $e^-$  off their atoms and cause pressure/collisional ionization. This typically occurs if atoms are closer than the normal size of a free atom. Hydrogen has effective size

$$a_H = 5.3 \times 10^{-9} \text{ cm} \quad \text{Bohr Radius}$$

So we expect pressure ionization of hydrogen when

$$\rho \gtrsim \frac{3m_p}{4\pi a_0^3} \sim 3 \text{ g/cm}^3$$

Although rare in stars, pressure ionization important in planetary interiors,

Partial ionization affects adiabatic index

$$\Gamma_1 = \left( \frac{d \ln P}{d \ln \rho} \right)_S = \frac{5}{3} \text{ for ideal gas}$$

When H partially ionized,  $\Gamma_1 \sim 1.2$ . The adiabatic index is  $\gamma \approx \Gamma_1 \sim 1.2$  in partial ionization zone. Recall convection occurs when

$$\nabla > \frac{\gamma-1}{\gamma} = \nabla_{\text{ad}}$$

So,  $\nabla_{\text{ad}}$  decreases in partial ionization zones, usually causing convection to occur. This happens at or just below the photospheres of many stars because stellar temperatures are similar to the ionization temperatures of H/He.

See state of matter diagram, HKT p. 172