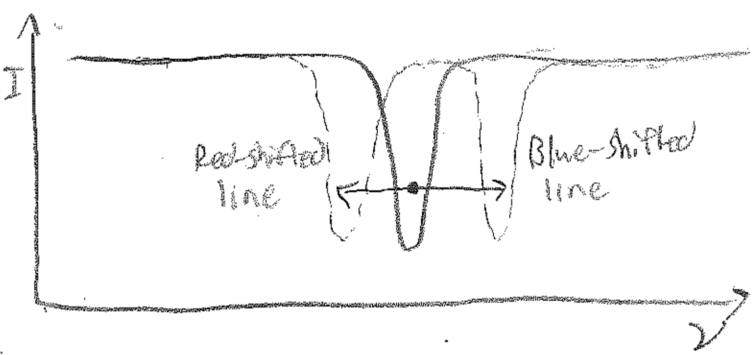
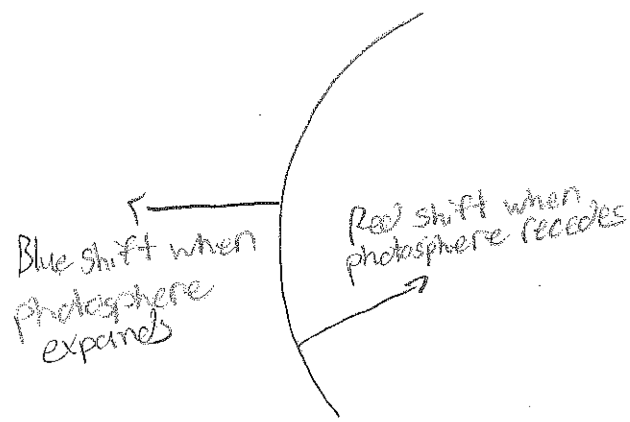
Ay 123 Lecture VII Helioseismology

Basic idea: stars pulsate as sound waves bounce between core and surface. A star is like a big stereo. We can observe pulsational motion at stellar surface. Because this pulsation is produced by waves traveling into interior of star, it carries information about the internal structure of the star.

How it's done: for the Sun, we measure the spectrum of the star, and look for Doppler shifts in absorption lines due to the radial velocity of the surface





For Sun, oscillations have periods $P \sim 5$ min (shorter than dynamical time scale!) and amplitudes $\Delta V_{phot} \sim 1$ m/s

Because oscillations occur on dynamical timescale ($t_{dyn} \ll t_{therm}$), oscillations are nearly adiabatic.

For adiabatic EOS,

$$P = K\rho^{\gamma}$$

$$\Rightarrow \quad \frac{\Delta P}{P} = \gamma \frac{\Delta \rho}{\rho}$$

ΔP, $\Delta \rho$ are Lagrangian perturbations of a fluid element that has been displaced from its equilibrium position.

$$\rho \rightarrow \rho_0 + \Delta\rho$$

$$P \rightarrow \underbrace{P_0}_{\text{equilibrium pressure}} + \Delta P \longrightarrow \text{Lagrangian perturbation}$$

$$r \rightarrow r_0 + \Delta r = r_0 + \xi_r \rightsquigarrow \text{Lagrangian Displacement}$$

Lagrangian perturbation: perturbation of fluid element

Eulerian perturbation: perturbation at fixed Eulerian position

Equations of hydrodynamics:

Continuity equation (conservation of mass)

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$

$\frac{d}{dt}$ ↳ Total time derivative

$\frac{\partial}{\partial t}$ → partial time derivative

Background has $\rho_0 = \rho_0(r)$, $\vec{v}_0 = 0$. Apply Lagrangian perturbation:

$$\boxed{\frac{\partial}{\partial t}\Delta\rho + \rho\vec{\nabla}\cdot\vec{v} = 0} \qquad (1)$$

where $\vec{v}$ is the perturbed velocity of the fluid element

The momentum equation ($\vec{F} = m\vec{a}$) is

$$\rho\frac{d}{dt}\vec{v} = -\vec{\nabla}P - \rho\vec{g}$$

To first order in perturbed quantities,

$$\rho \frac{\partial}{\partial t} \vec{v} = -\vec{\nabla}(\Delta P) + \frac{\xi_r}{r} \vec{\nabla} P - \Delta\rho\, \vec{g} - \rho \Delta \vec{g} \qquad (2)$$

Where $\vec{\xi}$ is the fluid displacement, and $\vec{v} = \frac{\partial}{\partial t} \vec{\xi}$.

Recall background has $\vec{v}_0 = 0$, $\vec{\nabla} P_0 = -\rho \vec{g}_0$ (hydrostatic equilib).

The perturbed gravitational field can be found from Poisson's equation. For a radial perturbation,

$$\frac{\Delta g}{g} = -2 \frac{\Delta r}{r} = -2 \frac{\xi_r}{r} \quad \text{(Radial perturbations only)}$$

Note that continuity equation + adiabatic eos can be combined:

$$\Rightarrow \frac{1}{\gamma P} \frac{\partial P}{\partial t} = -\vec{\nabla} \cdot \vec{v}$$

Let's now specify to the case of pressure waves, for which perturbed pressure ΔP is the main restoring force in equation (2). Taking the divergence of this equation,

$$\vec{\nabla} \cdot \left(\rho \frac{\partial}{\partial t} \vec{v} \right) = -\nabla^2 (\Delta P)$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot (\rho \vec{v}) \right) = -\nabla^2 (\Delta P)$$

We make a WKB approximation by only keeping spacial gradients of perturbed quantities. Then

$$\frac{\partial}{\partial t} \left(\rho \vec{\nabla} \cdot \vec{v} \right) = -\nabla^2 (\Delta P)$$

$$\Rightarrow -\frac{\partial^2}{\partial t^2} \Delta\rho = -\nabla^2 (\Delta P)$$

$$\Rightarrow \frac{\rho}{\gamma P} \frac{\partial^2}{\partial t^2} \Delta P = \nabla^2 (\Delta P)$$

This is a wave equation for ΔP. The solution is

$$\Delta P = A e^{i(\underbrace{\vec{k}}_{\text{wavenumber}} \cdot \vec{x} - \underbrace{\omega}_{\text{wave frequency}} t)}$$

where A is a constant. Inserting this in, we find

$$-\frac{\rho}{\gamma P} \omega^2 A = -k^2 A$$

Amplitude A cancels out. The factor $\underbrace{\frac{\gamma P}{\rho} = c_s^2}_{\text{Sound speed}}$

~mean particle speed in gas

So we have

$$\boxed{\omega^2 = k^2 c_s^2}$$

Acoustic wave dispersion relation

The wave vector is

$$k^2 = \underbrace{k_r^2}_{\text{Radial wave number}} + \underbrace{k_\perp^2}_{\text{Horizontal wavenumber}}$$

where $k_\perp^2 = \frac{\ell(\ell+1)}{r^2}$, ℓ = spherical harmonic index

So

$$\omega^2 = \left(k_r^2 + \frac{\ell(\ell+1)}{r^2}\right) c_s^2$$

Acoustic waves propagate where $k_r^2 > 0$. Thus, they propagate inward through the star until the point where

$$\omega^2 = L_\ell^2 = \frac{\ell(\ell+1)}{r^2} c_s^2$$

At inner turning point

WKB approximation breaks down near surface of star, because it is only valid where

$$K_r \gg \underbrace{\frac{1}{H}}_{\text{scale height}}$$

At stellar surface, H becomes small. Keeping gradients of stellar properties, dispersion relation becomes

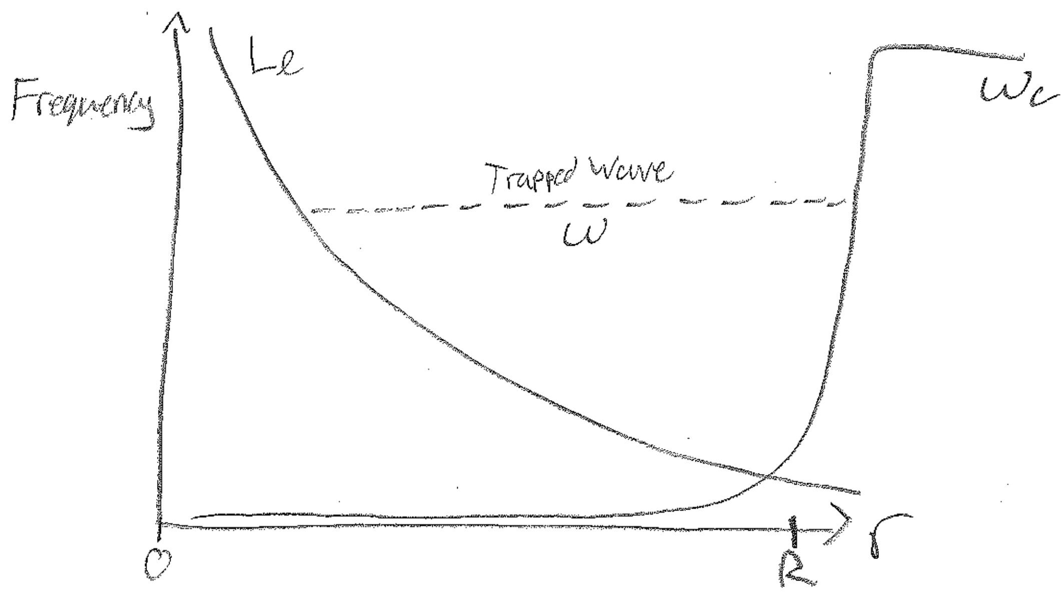
$$\omega^2 \simeq \left(K_r^2 + \underbrace{\frac{\ell(\ell+1)}{r^2}}_{\text{Large in interior}} + \underbrace{\left(\frac{1}{2H}\right)^2}_{\text{Large at surface}} \right) c_s^2$$

Defining $\omega_c = \frac{c_s}{2H}$ = Acoustic cut-off frequency

we have

$$(\omega^2 - \omega_c^2) \simeq \left(K_r^2 + \frac{\ell(\ell+1)}{r^2}\right) c_s^2$$

Wave propagates where K_r^2 is positive, become evanescent, i.e, it reflects, where K_r^2 is negative. This typically happens in core as $r \to 0$, and at surface where $H \to 0$. So, we get trapped waves, which form standing waves, i.e, stellar oscillation modes.



Angular Dependence

For spherical stars, modes have angular dependence

$$\Delta P, \Delta \rho, \xi_r \propto Y_{\ell m}(\theta, \phi)$$

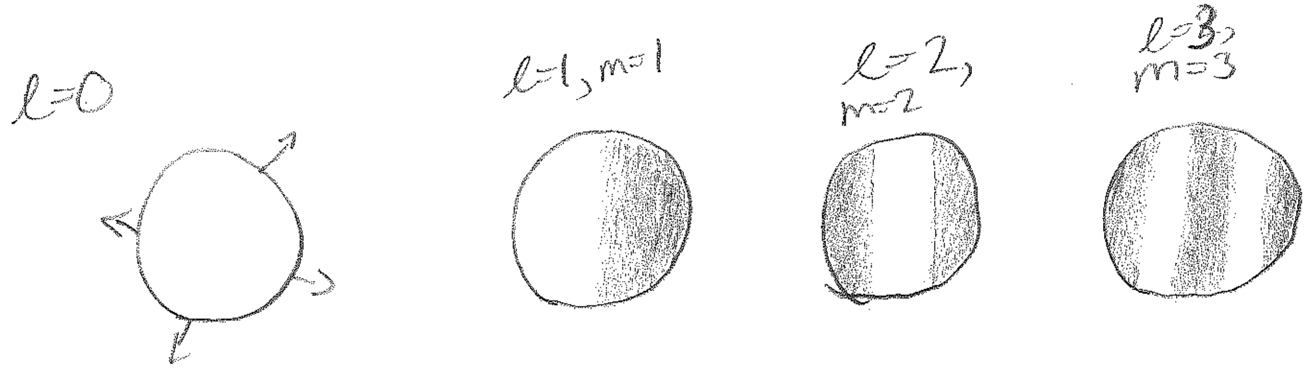
↳ Spherical Harmonic

Indexed by integers ℓ, m

ℓ → Angular Wavenumber

m → Azimuthal Wavenumber

$$-\ell \leq m \leq \ell$$



- In Sun, we can see up to $\ell \sim 2000$ because we can resolve surface. In other stars, we can only see up to $\ell \sim 3$. Above that, surface perturbations cancel out in disk-averaged observations, become hard to detect

Radial dependence: $\Delta P, \Delta \rho, \xi_r$, have radial dependence $\propto \sin(k_r r)$. Number of zero crossings is "radial eigenvalue" n. The eigenvalues n, ℓ, m tell us identity of mode.