Ay123 Set 3 solutions

Mia de los Reyes

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1. **Equation of state and the Chandrasekhar mass**

   (a) **Using the Fermi-Dirac distribution for non-relativistic electrons, derive the relationship between density and pressure, and hence appropriate value of γ and K.**

   The Fermi-Dirac distribution is given by

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \mathrm{d}p\; p^2 \left[\exp\left(\frac{\epsilon-\mu}{k_B T}\right)+1\right]^{-1} \tag{1}$$

   Then pressure is given by

$$P_e = \frac{1}{3}\int vp\frac{\mathrm{d}N}{\mathrm{d}^3x\mathrm{d}^3p}\mathrm{d}^3p \tag{2}$$

$$= \frac{8\pi}{3h^3}\int_0^\infty \mathrm{d}p\; vp^3 \left[\exp\left(\frac{\epsilon-\mu}{k_B T}\right)+1\right]^{-1} \tag{3}$$

   We can rewrite this in terms of momentum using the fact that the Fermi-Dirac distribution is essentially a step function:

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} \mathrm{d}p\; vp^3 \tag{4}$$

   where $p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$ is the Fermi momentum. In the non-relativistic limit, substitute $v \sim p/m_e$ and the definition of the Fermi momentum into equation (4):

$$P_e = \frac{8\pi}{3h^3 m_e}\int_0^{p_F} \mathrm{d}p\; p^4 \tag{5}$$

$$= \frac{8\pi}{3h^3 m_e}\frac{p_F^5}{5} \tag{6}$$

   Note that electron number density $n_e = \frac{\rho}{\mu_e m_p}$, where $\mu_e = 2$ for ^{4}He. Then we can write the electron pressure as

$$\boxed{P_e = \left(\frac{8\pi}{3h^3}\right)^{-2/3}\left(\frac{1}{5m_e}\right)\left(\frac{1}{\mu_e m_p}\right)^{5/3}\rho^{5/3}} \tag{7}$$

   Then $\boxed{\gamma = 5/3}$ and $\boxed{K = \left(\frac{8\pi}{3h^3}\right)^{-2/3}\left(\frac{1}{5m_e}\right)\left(\frac{1}{\mu_e m_p}\right)^{5/3} = 3.135\times 10^{12}\ \mathrm{cm}^4\mathrm{g}^{-2/3}\mathrm{s}^{-2}}$ in the polytropic equation $P = K\rho^\gamma$.

   (b) **Using the mass-radius relations for polytropes, derive the mass-radius relation for a white dwarf. Calculate the radius of a 1 $\mathrm{M}_\odot$ white dwarf.**

   From the discussion of polytropes (HKT Eq. 7.40):

$$K = \left[\frac{4\pi}{\xi^{n+1}(-\theta_n)^{n-1}}\right]_{\xi_1}^{1/n}\frac{G}{n+1}M^{1-1/n}R^{-1+3/n} \tag{8}$$

Here, $n = \frac{1}{\gamma-1} = 3/2$ for a $\gamma = 5/3$ polytrope. From HKT Table 7.1, $\xi_1 = 3.6538$ and $-\theta_n(\xi_1) = 0.20330$ for $n = 3/2$. Set this equal to the equation for K from part (a) and solve for R: $\boxed{R = (1.11 \times 10^{20}\ \mathrm{cm\ g}^{1/3})\ \mathrm{M}^{-1/3}}$.

For a 1 $\mathrm{M}_\odot$ white dwarf, this yields $\boxed{R = 8.81 \times 10^8\ \mathrm{cm} \approx 0.01\ \mathrm{R}_\odot}$.

(c) **Now assume the helium white dwarf is supported by highly relativistic degeneracy pressure. Use the Fermi-Dirac distribution to derive the appropriate values of γ and K, and then derive its mass. Express the mass in units of $\mathrm{M}_\odot$.**

Now consider relativistic electrons, so $v = c$. Then

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} \mathrm{d}p\ vp^3 \tag{9}$$

$$= \frac{8\pi c}{3h^3}\int_0^{p_F} \mathrm{d}p\ p^3 \tag{10}$$

$$= \frac{8\pi c}{3h^3}\frac{p_F^4}{4} \tag{11}$$

$$= \frac{8\pi c}{3h^3}\frac{c}{4}\left(\frac{3h^3 n_e}{8\pi}\right)^{4/3} \tag{12}$$

Again, plugging in the definition for $n_e = \frac{\rho}{\mu_e m_p}$, we find that the pressure goes as

$$P_e = \left(\frac{8\pi}{3h^3}\right)^{-1/3}\left(\frac{c}{4}\right)\left(\frac{1}{\mu_e m_p}\right)^{4/3}\rho^{4/3} \tag{13}$$

So $\boxed{\gamma = 4/3}$ and $\boxed{K = \left(\frac{8\pi}{3h^3}\right)^{-1/3}\left(\frac{c}{4}\right)\left(\frac{1}{\mu_e m_p}\right)^{4/3} = 4.9 \times 10^{14}\ \mathrm{g}^{-1/3}\ \mathrm{cm}^3\ \mathrm{s}^{-2}}$ (again assuming $\mu_e = 2$ for ^{4}He).

To get the mass, start with the equation given in class for a polytrope:

$$M = 4\pi\left(\frac{n+1}{4\pi G}K\right)^{3/2}\rho_c\frac{3-n}{2n}\left(-\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi_1} \tag{14}$$

where $n = \frac{1}{\gamma-1} = 3$ for a $\gamma = 4/3$ polytrope. Then

$$M = 4\pi\left(\frac{K}{\pi G}\right)^{3/2}(\xi_1)^2(-\theta_n)_{\xi_1} \tag{15}$$

Use values from HKT Table 7.1: $\xi_1 = 6.8969$ and $-\theta_n(\xi_1) = -.04243$ for $n = 3$ to find $\boxed{M = 1.43\ \mathrm{M}_\odot}$. This is very close to the Chandrasekhar mass!

(d) **Recalculate this mass for a relativistic white dwarf made of pure gold (^{197}Au). Gold is currently worth $39.47/g. What is the value of this golden dwarf?**

The only thing that changes here is the value of μ_e, which is the number of baryons per electron. For ^{197}Au, which has atomic number $Z = 79$, this yields $\mu_e = \frac{197}{79} = 2.49$.

Then use the formula from part (c) to find $K = 3.7 \times 10^{14}\ \mathrm{g}^{-1/3}\ \mathrm{cm}^3\ \mathrm{s}^{-2}$. Plug this into equation (15) for M to find $\boxed{M = 1.88 \times 10^{33}\ \mathrm{g} = 0.94\ \mathrm{M}_\odot.}$

The cost of this golden dwarf is $\boxed{\$7.42 \times 10^{34}}$!

2. **Nuclides and kilonova event rates**

(a) **In the Sun, the mass fraction of r-process nuclides is $X_{rp} \sim 10^{-7}$. The stellar mass of the Milky Way is $M_{\mathrm{MW}} \sim 10^{11}\ \mathrm{M}_\odot$. Assuming similar abundances in other stars, estimate the total r-process mass within the Milky Way.**

$M_{rp,\mathrm{MW}} = X_{rp}M_{\mathrm{MW}} = \boxed{10^4\ \mathrm{M}_\odot}$.

(b) **Assuming r-process elements are produced solely in neutron star mergers, and that $M_{rp} \sim 0.03$ $\mathrm{M}_\odot$ of r-process nuclides were expelled from GW170817, estimate the number of neutron star mergers that have occurred in the Milky Way, and the average neutron star merger rate over the $\tau_{\mathrm{MW}} \sim 10$ Gyr lifetime of the Milky Way.**

The number of neutron star mergers is given by N_{NSM}=(mass of r-process elements)/(mass of r-process elements per NSM). So $N_{\mathrm{NSM}} = \frac{M_{rp,\mathrm{MW}}}{M_{rp}} = \boxed{3.3 \times 10^5}$.

The average rate of neutron star mergers is just $N_{\mathrm{NSM}}/\tau_{\mathrm{MW}} = \boxed{3.3 \times 10^{-5}\ \mathrm{yr}^{-1}}$.

(c) **Type Ia supernovae synthesize $M_{\mathbf{Fe}} \sim 0.5$ $\mathrm{M}_\odot$ of iron, whose mass fraction in the Sun is $X_{\mathbf{Fe}} \sim 10^{-3}$. Estimate the average Type Ia supernovae rate of the Milky Way.**

As in part (b), first compute the total mass of iron in the Milky Way: $M_{\mathrm{Fe,MW}} = 10^8$ $\mathrm{M}_\odot$. Then follow the procedure in part (c) to compute the number of Type Ia SNe: $N_{\mathrm{Ia}} = \frac{M_{\mathrm{Fe,MW}}}{M_{\mathrm{Fe}}} = 2 \times 10^8$. Divide this by the lifetime of the Milky Way to get the average Type Ia SNe rate: $N_{\mathrm{Ia}}/\tau_{\mathrm{MW}} = \boxed{0.02\ \mathrm{yr}^{-1}}$.

(d) **Consider a dwarf galaxy with iron abundance 10^{-1} that of the Milky Way, and stellar mass $M_{\mathrm{gal}} = 10^6\ M_\odot$. Assuming the iron to r-process abundance ratio is the same as the Milky Way, estimate the total mass of iron and r-process elements in the dwarf galaxy.**

First compute the total mass of iron in the dwarf galaxy: $M_{\mathrm{Fe,gal}} = 10^{-1} X_{\mathrm{Fe,MW}} M_{\mathrm{gal}} = \boxed{10^2\ \mathrm{M}_\odot}$.

The ratio of iron to r-process abundance in the Milky Way is: $\frac{M_{\mathrm{Fe}}}{M_{\mathrm{rp}}} = \frac{10^8\ \mathrm{M}_\odot}{10^4\ \mathrm{M}_\odot} = 10^4$.

Assuming this is the same in the dwarf galaxy, we can compute the total mass of r-process elements in the dwarf galaxy: $M_{rp,\mathrm{gal}} = M_{\mathrm{Fe,gal}} \left(\frac{M_{\mathrm{Fe}}}{M_{\mathrm{rp}}}\right)^{-1} = 10^2\ \mathrm{M}_\odot (10^4)^{-1} = \boxed{10^{-2}\ \mathrm{M}_\odot}$.

(e) **What is the expected number of type Ia supernovae that have occurred in the dwarf galaxy? What is the estimated number of neutron star mergers in the dwarf galaxy? Do we expect all dwarf galaxies to be enriched in r-process elements such as Europium?**

As in parts (b) and (c), compute the number of Type Ia SNe and neutron star mergers:

$N_{\mathrm{Ia}} = \frac{M_{\mathrm{Fe,gal}}}{M_{\mathrm{Fe}}} = \boxed{200}$

$N_{\mathrm{NSM}} = \frac{M_{rp,\mathrm{gal}}}{M_{rp}} = \boxed{0.33}$

We do not expect all dwarf galaxies to be enriched in r-process elements.