Ay101 Set 2 solutions

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October 19 2019

1. **Convection**

(a) **If a convective blob accelerates from zero velocity over a mixing length Λ according to $\frac{\mathrm{d}}{\mathrm{d}r}v_{\mathrm{con}} = |N|$, find the maximum convective velocity v_{con} in terms of N and the mixing length Λ. Express v_{con} in terms of α, γ, $(\nabla_{\mathrm{ad}} - \nabla)$, ρ, and c_s.**

We want to solve the differential equation $\frac{\mathrm{d}}{\mathrm{d}r}v_{\mathrm{con}} = |N|$ to get $v_{\mathrm{con}}(r)$. This is pretty straightforward, since we assume N is constant over a mixing length, so we find:

$$v_{\mathrm{con}}(r) = |N|r + C \tag{1}$$

To solve for the integration constant, we use the boundary condition $v_{\mathrm{con}}(r = 0) = 0$. This conveniently yields $C = 0$.

Now we want the maximum v_{con}, which should occur at the mixing length $r = \Lambda$. The maximum convective velocity should therefore be $\boxed{v_{\mathrm{con}} = |N|\Lambda}$. We can also rewrite this by plugging in $\Lambda = \alpha H$, $N^2 = \frac{g}{H}(\nabla_{\mathrm{ad}} - \nabla)$, and $H = P/(\rho g)$:

$$\begin{aligned} v_{\mathrm{con}} &= |N|\Lambda && (2)\\ &= \left(\frac{g}{H}\right)^{1/2}(\nabla_{\mathrm{ad}} - \nabla)^{1/2}\alpha H && (3)\\ &= \left(\frac{P}{\rho}\right)^{1/2}(\nabla_{\mathrm{ad}} - \nabla)^{1/2}\alpha && (4)\end{aligned}$$

Recall that a polytropic equation of state is given by $P = \rho^{\gamma}$, and the sound speed is given by $c_s^2 = \gamma P/\rho$. So $(P/\rho)^{1/2} = \sqrt{\rho^{\gamma-1}}$, and the sound speed is $c_s = \sqrt{\gamma\rho^{\gamma-1}}$.

Plugging this into equation (4) yields $\boxed{v_{\mathrm{con}} = \gamma^{-1/2}c_s(\nabla_{\mathrm{ad}} - \nabla)^{1/2}\alpha}$.

(b) **Express the kinetic energy flux $F_{\mathrm{con}} = \rho v_{\mathrm{con}}^3$ of upgoing convective blobs.**

Substituting v_{con} from (a), the convective flux is $\boxed{F_{\mathrm{con}} = \rho\gamma^{-3/2}(\alpha c_s)^3(\nabla_{\mathrm{ad}} - \nabla)^{3/2}}$.

(c) **For $\alpha = 2$, what is the value of $(\nabla_{\mathrm{ad}} - \nabla)$ required for convection to carry the Sun's luminosity? What is the corresponding maximum convective velocity v_{con}, and how does this compare to the sound speed c_s?**

The energy flux going through the base of the convective zone r is $F = \frac{L_\odot}{4\pi r^2}$. Setting this equal to the convective energy flux F_{con} from part (b), we can solve for $(\nabla_{\mathrm{ad}} - \nabla)$:

$$\begin{aligned} \rho\gamma^{-3/2}(\alpha c_s)^3(\nabla_{\mathrm{ad}} - \nabla)^{3/2} &= \frac{L_\odot}{4\pi r^2} && (5)\\ (\nabla_{\mathrm{ad}} - \nabla) &= \left(\frac{L_\odot}{\rho 4\pi r^2}\right)^{2/3}\gamma(\alpha c_s)^{-2} && (6)\end{aligned}$$

Substituting the given values (note that $\gamma = 5/3$ for an ideal gas and $L_\odot = 4 \times 10^{33}$ erg/s is a good value to remember), we find $\boxed{(\nabla_{\mathrm{ad}} - \nabla) = 1.2 \times 10^{-7}}$.

Using the expression from part (a), this corresponds to $\boxed{v_{\mathrm{con}} = 1.1 \times 10^4 \text{ cm/s}}$, which is much slower than the sound speed $c_s = 2 \times 10^7$ cm/s.

(d) **Assuming convection carries all the Sun's luminosity, use the expression for N^2 to find the density gradient in a convection zone.**

From the given equation for N^2, we know that

$$g\left[\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}+\frac{g}{c_s^2}\right]=-N^2 \tag{7}$$

We can solve this for $\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}$. Plugging in $N^2=\frac{g}{H}(\nabla_{\mathrm{ad}}-\nabla)$, $H=P/(\rho g)$, and equation (6) for $(\nabla_{\mathrm{ad}}-\nabla)$, we find

$$\begin{aligned}\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} &= \frac{-N^2}{g}-\frac{g}{c_s^2} & (8)\\ &= -\frac{1}{H}(\nabla_{\mathrm{ad}}-\nabla)-\frac{g}{c_s^2} & (9)\\ &= -\frac{\rho g}{P}\left(\frac{L_\odot}{4\pi r^2\rho}\right)^{2/3}\frac{\gamma}{\alpha^2 c_s^2}-\frac{g}{c_s^2} & (10)\end{aligned}$$

Now remember that we're using a polytropic equation of state, so $c_s^2=\gamma P/\rho$. Substituting this into equation (10), we find

$$\boxed{\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}=-\frac{g}{c_s^2}\left[\left(\frac{L_\odot}{4\pi r^2\rho c_s^3}\right)^{2/3}\frac{\gamma^2}{\alpha^2}+1\right]} \tag{11}$$

as expected.

For the solar values given in part (c), we find that $\boxed{\left(\frac{L_\odot}{4\pi r^2\rho c_s^3}\right)^{2/3}\frac{\gamma^2}{\alpha^2}=2\times10^{-7}}$, which is much smaller than 1. The first term in equation (11) can then be treated as negligible, so the density gradient becomes $\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}=\frac{g}{c_s^2}$. This suggests that the Sun's density profile is not strongly dependent on α at the convective zone.

2. **Fully convective cool stars**

(a) **The polytropic convective envelope extends nearly all the way to the photosphere. Use this to derive a scaling between pressure and temperature at the photosphere.**

The gas is a polytrope, so $P=K\rho^\gamma$. We want to get rid of ρ in this expression by writing it in terms of P and T. The envelope is an ideal gas, so $P=\frac{\rho k_B T}{\mu m_H}$. This means that $\rho=\frac{P\mu m_H}{k_B T}\propto PT^{-1}$. We can plug this scaling into the polytropic equation to write P as a function of T:

$$P\propto P^\gamma T^{-\gamma} \tag{12}$$

$$P^{1-\gamma}\propto T^{-\gamma} \tag{13}$$

$$\boxed{P\propto T^{\frac{\gamma}{\gamma-1}}} \tag{14}$$

(b) **Use the fact that H- opacity dominates in cool stars and has the scaling $\kappa_{H-}\propto T^9$, the fact that $P_s=\frac{2g_s}{3\kappa_s}$, and the result from part (a) to determine a scaling between M, R, and T_{eff} for cool stars.**

Set the results of parts (b) and (c) equal and plug in $g\propto MR^{-2}$ and $\kappa\propto T^9$. Keep the proportionality constant in the polytropic equation of state, since it turns out this constant (we'll call it K, as in $P=KT^{\frac{\gamma}{\gamma-1}}$) depends on M and R:

$$KT^{\frac{\gamma}{\gamma-1}}\propto\frac{g_s}{\kappa_s} \tag{15}$$

$$KT^{\frac{\gamma}{\gamma-1}}\propto MR^{-2}T^{-9} \tag{16}$$

$$MR^{-2}\propto KT^{9+\frac{\gamma}{\gamma-1}} \tag{17}$$

We know $K = M^{-1/2}R^{-3/2}$, so we can plug this expression for K into equation (29) to get $\boxed{M^{3/2}R^{-1/2} \propto T^{9+\frac{\gamma}{\gamma-1}}}$. For an ideal gas of $\gamma = 5/3$, this is $\boxed{M^{3/2}R^{-1/2} \propto T^{23/2}}$.

(c) **Main sequence G/K/M stars have $L \propto M^3$. Add the lower part of the main sequence to the HR diagram.**

Relate L, M, and T using the fact that for a blackbody, $L \propto R^2T^4$. First relate L, M, and R using the solution from part (d):

$$T \propto L^{1/4}R^{-1/2} \Rightarrow M^{3/2}R^{-1/2} \propto L^{23/8}R^{-23/4} \Rightarrow L^{23/8} \propto M^{3/2}R^{21/4} \tag{18}$$

Then get rid of R using $R \propto L^{1/2}T^{-2}$:

$$L^{23/8} \propto M^{3/2}L^{21/8}T^{-21/2} \Rightarrow L^{-1/4}M^{3/2} \propto T^{21/2} \tag{19}$$

Instead of treating M fixed, we'll plug the scaling relation for $M \propto L^{1/3}$ into equation (36) to find $L^{-1/4}L^{1/2} \propto T^{21/2}$, which yields $\boxed{L \propto T^{42}}$.

(d) **Draw an evolutionary track for a red giant branch star on an HR diagram.**

Since M is fixed for a given star, equation (19) yields $\boxed{L \propto T^{-42}}$, a near vertical track on the HR diagram.

3. **Field Color Magnitude Diagram**

See attached Jupyter notebook `Ay101_ps2_3.ipynb` for one solution.

4. **Evolution Tracks with MESA**