Ay101 Set 1 solutions

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1. **The scale of the Sun (10 pts total)**

   (a) **(2 pts) Venus has an orbital period of 225 days. Using Kepler's laws, what is the semi-major axis of Venus (in AU)?**
   Start with Kepler's third law:

$$\Omega^2 = \frac{G(M_\odot + M)}{a^3} \tag{1}$$

First, consider the Earth-Sun system, assuming that $M_E \ll M_\odot$:

$$\Omega_E^2 = \frac{GM_\odot}{a_E^3} \tag{2}$$

Now, Ω_E can be calculated from known values (Earth's period is 1 year) and a_E is known (it's 1 AU), so can solve for $M_\odot$:

$$M_\odot = \frac{\Omega_E^2 a_E^3}{G} \tag{3}$$

Then consider the Venus-Sun system, again assuming that $M_V \ll M_\odot$:

$$\Omega_V^2 = \frac{GM_\odot}{a_V^3} \tag{4}$$

Then solve for a_V, substituting equation (9) for $M_\odot$ and converting angular velocities to periods ($P = \frac{2\pi}{\Omega}$):

$$a_V = \left(\frac{GM_\odot}{\Omega_V^2}\right)^{1/3} \tag{5}$$

$$= \left(\frac{\Omega_E^2 a_E^3}{\Omega_V^2}\right)^{1/3} \tag{6}$$

$$= \left(\frac{P_V}{P_E}\right)^{2/3} a_E \tag{7}$$

Plug in known values ($P_E = 365.25$ days, $a_E = 1$ AU) and given values ($P_V = 225$ days), and find that $\boxed{a_V = 0.724 \text{ AU}}$.

   (b) **(2 pts) At conjunction with the Sun, it takes astronomers on Earth 276s to detect the radio waves that reflect off Venus. Assuming circular orbits for the Earth and Venus, compute the distance in 1 AU in cgs units.**
   Note: "conjunction" = Venus is directly between Earth and the Sun. Call the distance between Earth and Venus d_V. The light takes time t to travel distance $2d_V$:

$$d_V = \frac{ct}{2} = 4.14 \times 10^{12} \text{ cm} \tag{8}$$

From the previous problem, we can compute d_V in terms of AU:

$$d_V = a_E - a_V = (1 - 0.724) \text{ AU} = 0.276 \text{ AU} \tag{9}$$

Then combine equations (14) and (15) to convert AU to cm: $\boxed{1 \text{ AU} = 1.5 \times 10^{13} \text{ cm}}$.

(c) **(2 pts each) Use your results above to compute the absolute mass of the Sun in cgs units. Using its measured angular radius of 16 arcmin, compute the radius of the Sun in cgs units.**

To find the mass, use Kepler's third law: $M_\odot = \frac{\Omega_E^2 a_E^3}{G}$. Plug in known values in cgs to get $\boxed{M_\odot = 2.0 \times 10^{33} \text{ g}}$.

To find the radius, use the definition of angular size: $R_\odot = a_E \sin\theta \approx a_E\theta$. Plug in known values in cgs to get $\boxed{R_\odot = 6.96 \times 10^{10} \text{ cm}}$.

(d) **(2 pts) The surface temperature of the Sun is $T = 5770$ K. Using the blackbody radiation law, compute the luminosity of the Sun in cgs units.**

From the Stefan-Boltzmann law for a blackbody:

$$L = 4\pi R^2 \sigma_B T_{\text{eff}}^4 \tag{10}$$

Plug in R from part (c) and known values and constants to get $\boxed{L_\odot = 3.83 \times 10^{33} \text{ erg/s}}$.

2. **Stars are gases (10 pts total)**

(a) **(6 pts) Provide a quantitative relation between the temperature and density of a star which indicates when we can treat it as a gas throughout its interior, in spite of the very high densities. Is our assumption valid at the center of the Sun, where the density is about 100 times the average density?**

To check if the center of the Sun can be treated as a gas, we can compare Coulomb energy to thermal energy. The ideal gas law is reasonable when the thermal energy ($\sim k_B T$) is larger than the Coulomb energy ($\sim (Ze^2)/r$). This occurs when

$$kT \gg Ze^2/r \tag{11}$$

$$r \gg Ze^2/kT \tag{12}$$

where r is the interparticle distance and Z is the charge of the ion ($Z = 1$ for a gas composed only of ionized hydrogen). By thinking of the number density n as r^{-3}, we can write r in terms of mass density: $r = n^{-1/3} = (\rho/\mu m_p)^{-1/3}$. Then we can rewrite equation (12):

$$\boxed{\rho \ll \mu m_p (k/e^2)^3 T^3} \tag{13}$$

For atomic hydrogen, $\mu = 0.5$. Plugging this in, our condition for treating a star as an ideal gas is (in cgs units):

$$\rho \ll 1.8 \times 10^{-16} T^3 \tag{14}$$

The Sun's central temperature is $T_c \sim 10^7$ K. By equation (19), we require $\boxed{\rho_c \ll 6 \times 10^5 \text{ g cm}^{-3}}$. Since the Sun's central density[1] ρ_c is only ~ 150 g cm^{-3}, we may treat the sun as an ideal gas throughout, and need not consider Coulomb interactions between particles.

(b) **(4 pts) If all stars have roughly the same central temperature, use a scaling argument to determine the stellar mass at which the ideal gas assumption breaks down.**

Now we want to know how ρ scales with stellar mass M. (In the following derivation we only care about approximate scaling arguments, so don't worry about prefactors.) The internal energy can be approximated as $U \propto \frac{k_B T}{\mu m_p} M$ at the central temperature T. Now recall that by the virial theorem, the internal energy is approximately the gravitational energy $\Omega \propto GM^2/R$. Solve for T and find that $T \propto M/R$.

We can then assume that all stars have roughly the same central temperature (which is actually a good approximation for main-sequence stars), so the central density $M \propto R$. Then $\rho \propto M/R^3 \propto M^{-2}$, so $M \propto \rho^{-1/2}$.

[1] An easy way to get this is by remembering that the Sun's average density is approximately that of water (~ 1 g cm^{-3}), and the Sun's central density is about 100 times this.

Plugging in values for the Sun, we find $M_{\rm lim} = \left(\frac{\rho_{\rm lim}}{\rho_\odot}\right)^{-1/2} M_\odot$. This yields a limiting mass of $\boxed{M = 0.016 M_\odot}$, which is about the mass of brown dwarfs or giant planets (not stars!). Therefore, we will never have to consider Coulomb interactions for main sequence stars.

3. **A toy star (15 pts total)** Assume that a star obeys the density model

$$\rho(r) = \rho_c \left(1 - \frac{r}{R}\right). \tag{15}$$

(a) **(4 pts) Find an expression for the central density in terms of R and M.**

Solve for total mass M by integrating over the density profile:

$$\begin{aligned} M &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^R r^2 \rho(r) \mathrm{d}r && (16) \\ &= 4\pi \int_0^R \rho_c \left(r^2 - \frac{r^3}{R}\right) \mathrm{d}r && (17) \\ &= 4\pi\rho_c \left(\frac{R^3}{3} - \frac{R^3}{4}\right) && (18) \\ &= \frac{\pi}{3}\rho_c R^3 && (19) \end{aligned}$$

Then solve for central density: $\boxed{\rho_c = \frac{3M}{\pi R^3}}$.

(b) **(5 pts) Use the equation of hydrostatic equilibrium and zero boundary conditions to find the pressure as a function of radius. What is the dependence of the central pressure P_c in terms of M and R?**

Doing the same integral as in the previous problem, we know that

$$m(r) = \frac{4\pi}{3}\rho_c r^3 \left(1 - \frac{3r}{4R}\right) \tag{20}$$

Now use the equation of hydrostatic equilibrium:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{Gm}{r^2}\rho(r) && (21) \\ &= -G\frac{4\pi}{3}\rho_c r \left(1 - \frac{3r}{4R}\right) \rho_c \left(1 - \frac{r}{R}\right) && (22) \\ &= -\frac{4\pi}{3} G\rho_c^2 r \left(1 - \frac{7r}{4R} + \frac{3r^2}{4R^2}\right) && (23) \end{aligned}$$

Integrate equation (23) to get the total pressure:

$$\begin{aligned} P(r) &= -\frac{4\pi}{3} G\rho_c^2 \int \left(r - \frac{7r^2}{4R} + \frac{3r^3}{4R^2}\right) \mathrm{d}r && (24) \\ &= -\frac{4\pi}{3} G\rho_c^2 \left[\frac{r^2}{2} - \frac{7r^3}{12R} + \frac{3r^4}{16R^2} + C\right] && (25) \end{aligned}$$

Use the zero boundary condition ($P(R) = 0$) to solve for the integration constant C:

$$\begin{aligned} P(R) &= -\frac{4\pi}{3} G\rho_c^2 R^2 \left[\frac{1}{2} - \frac{7}{12} + \frac{3}{16} + \frac{C}{R^2}\right] = 0 && (26) \\ C &= R^2 \left(-\frac{1}{2} + \frac{7}{12} - \frac{3}{16}\right) && (27) \\ &= -\frac{5}{48} R^2 && (28) \end{aligned}$$

Now let's solve for the central pressure $P_c = P(r=0)$:

$$P_c = -\frac{4\pi}{3} G\rho_c^2 C \tag{29}$$
$$= \frac{5\pi}{36} G\rho_c^2 R^2 \tag{30}$$

Okay, finally we can substitute stuff into equation (30) to write the full equation for pressure:

$$P(r) = -\frac{4\pi}{3} G\rho_c^2 \left[\frac{r^2}{2} - \frac{7r^3}{12R} + \frac{3r^4}{16R^2} - \frac{5}{48}R^2\right] \tag{31}$$
$$= P_c \left[1 - \frac{24}{5}\left(\frac{r}{R}\right)^2 + \frac{28}{5}\left(\frac{r}{R}\right)^3 - \frac{9}{5}\left(\frac{r}{R}\right)^4\right] \tag{32}$$

So we find that $\boxed{P(r) = P_c \times f\left(\frac{r}{R}\right)}$ as expected.

Now plug in the answer for part (a) to get P_c as a function of M and R:

$$P_c = \frac{5\pi}{36} G \left(\frac{3M}{\pi R^3}\right)^2 R^2 \tag{33}$$

We can simplify this to $\boxed{P_c = \frac{5}{4\pi}\frac{GM^2}{R^4}}$.

(c) **(3 pts) What is the central temperature T_c, assuming an ideal gas equation of state? How does it scale with mean particle mass?**

Ideal gas: $P = \frac{\rho k_B T}{\mu m_p}$

Solve this for central temperature, plugging in answer from (b) for P_c and answer from (a) for ρ_c:

$$T_c = \frac{P_c \mu m_p}{\rho_c k_B} \tag{34}$$
$$= \frac{\frac{5\pi}{36} G\rho_c R^2 \mu m_p}{k_B} \tag{35}$$
$$= \frac{\frac{5\pi}{36} G \frac{3M}{\pi R^3} R^2 \mu m_p}{k_B} \tag{36}$$

This simplifies to $\boxed{T_c = \frac{5GM}{12R}\frac{\mu m_p}{k_B}}$ which scales as $\boxed{T_c \propto \mu m_p}$. ($\mu m_p$ is the mean particle mass.)

(d) **(3 pts) Find the ratio of radiation pressure to gas pressure at the center of the star as a function of total stellar mass in $M_\odot$. At what mass does radiation pressure become comparable to the ideal gas pressure?**

Radiation pressure is given by $P_{\rm rad} = \frac{1}{3}a_o T^4$. The ratio at the center of the star is therefore

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{1}{3}a_o \frac{T_c^4}{\frac{5}{4\pi}\frac{GM^2}{R^4}} \tag{37}$$

Then plug in T_c from part (c):

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{1}{3}a_o \frac{\left(\frac{5GM}{12R}\frac{\mu m_p}{k_B}\right)^4}{\frac{5}{4\pi}\frac{GM^2}{R^4}} \tag{38}$$
$$= \frac{125\pi}{15552} a_o G^3 M^2 \left(\frac{\mu m_p}{k_B}\right)^4 \tag{39}$$

Assuming solar composition ($\mu = 0.62$), we can rewrite this in terms of solar masses as:

$\boxed{\frac{P_{\rm rad}}{P_{\rm gas}} = 7.2 \times 10^{-4}\left(\frac{M}{M_\odot}\right)^2}$ When $\frac{P_{\rm rad}}{P_{\rm gas}} = 1$, the mass of the star is $\boxed{M \approx 37 M_\odot}$. Note that this is not an exact result, since our formula for T_c assumes that radiation pressure is negligible.

4. **Cluster color magnitude diagram (15 pts total)**

   The goal of this problem is to make a color magnitude diagram for M67, a nearby open cluster, using Gaia data.

   See the provided Jupyter notebook "Ay101_ps1_4.ipynb" for one potential solution.

5. **Using the MESA stellar evolution code (10 pts)**

   (a) **(4 pts) Run the default stellar model located in mesa/star/work/. At time step 10, what is the core temperature and surface temperature of the model?**

   The core temperature is $\log T_c = 5.48$ [K], or $\boxed{T_c = 3.012 \times 10^5 \text{ K}}$. The surface temperature is $\boxed{T_{\text{eff}} = 3452 \text{ K}}$.

   (b) **(6 pts) Evolve a $1M_\odot$ model up the red giant branch. What is the surface temperature and radius of the star when its luminosity reaches $50L_\odot$?**