Ay101 Problem Set 3

due Monday, November 11

1. **Equation of state and the Chandrasekhar Mass** (10 points)

   Consider a white dwarf composed of electrically neutral gas, consisting of helium ions and fully degenerate electrons. Assume that the ions contribute negligibly to the pressure and that the gas obeys a polytropic equation of state, $P = K\rho^{\gamma}$.

   (a) Using the Fermi-Dirac distribution for non-relativistic electrons, derive the relationship between density and pressure, and hence appropriate value of γ and K.

   It can be shown (see HKT Eq. 7.40) that the mass-radius relation of a polytropic star is given by the relation

$$K = \left[\frac{4\pi}{\xi_1^{n+1}(-\theta_1)^{n-1}}\right]^{1/n} \frac{G}{n+1} M^{1-1/n} R^{-1+3/n}\,, \tag{1}$$

   where $n = 1/(\gamma - 1)$, and $\xi_1 = 3.6538$ and $-\theta_1 = 0.20330$ for a $n = 3/2$ polytrope.

   (b) Using the polytropic relations above and your result from part b, derive the mass-radius relation for a white dwarf. Calculate the radius of a $1\,M_{\odot}$ white dwarf.

   (c) Now assume the helium white dwarf is supported by highly relativistic degeneracy pressure. Use the Fermi-Dirac distribution to derive the appropriate values of γ and K, and then derive its mass. From HKT, $\xi_1 = 6.8969$ and $-\theta_1 = 0.04243$ for a $n = 3$ polytrope. Express the mass in units of $M_{\odot}$.

   (d) Recalculate this mass for a relativistic white dwarf made of pure gold (^{197}Au). Gold is currently worth 47.68 \$/g. What is the value of this golden dwarf?

2. **Nuclides and Kilonova Event Rates** (10 points)

   (a) In the Sun, the mass fraction of r-process nuclides is $X_{rp} \sim 10^{-7}$. The stellar mass of the Milky Way is $M_{\rm MW} \sim 10^{11}\,M_{\odot}$. Assuming similar abundances in other stars, estimate the total r-process mass within the Milky Way.

   (b) From observations of GW170817, it is estimated that $M_{rp} \sim 0.03\,M_{\odot}$ of r-process nuclides were expelled from the kilonova. Assuming r-process elements are produced solely in neutron star mergers, estimate the number of neutron star mergers that have occurred in the Milky Way, and the average neutron star merger rate over the $T_{\rm MW} \sim 10\,$Gyr lifetime of the Milky Way.

   (c) Type Ia supernovae synthesize $M_{\rm Fe} \sim 0.5\,M_{\odot}$ of iron, whose mass fraction in the Sun is $X_{\rm Fe} \sim 10^{-3}$. Estimate the average Type Ia supernovae rate of the Milky Way.

   (d) Consider a dwarf galaxy with iron abundance 10^{-1} that of the Milky Way, and stellar mass $M_{\rm gal} = 10^{6}\,M_{\odot}$. Assuming the iron to r-process abundance ratio is the same as the Milky Way, estimate the total mass of iron and r-process elements in the dwarf galaxy.

   (e) What is the expected number of type Ia supernovae that have occurred in the dwarf galaxy? What is the estimated number of neutron star mergers in the dwarf galaxy? Do we expect all dwarf galaxies to be enriched in r-process elements such as Europium?

3. **Cluster Age and Evolution** (10 points)

   From the *Gaia* database, download stars that correspond to the Pleiades, Praesepe, and Hyades clusters. For the Pleiades and Praesepe, you'll want to search out to at least 4 degrees, and for the Hyades out to at least 8 degrees. To avoid downloading too many background stars, I'd recommend only selecting stars with parallax greater than 3 milliarcseconds for Pleiades and Praesepe, and greater than 10 milliarcseconds for Hyades. Use the advanced search query to increase the maximum number of stars that you download.

   (a) As in homework 1, select cluster stars using parallax and proper motion cuts (these can be done by eye). Make a color-magnitude diagram (make sure to plot absolute magnitude), with stars in each cluster shaded by a different color or symbol. Is reddening very important for these clusters? Identify the main sequence turnoff, red giants, and white dwarfs. Which cluster is youngest, and how do you know?

   (b) Estimate the absolute magnitude M_g for stars at the main sequence turnoff in each cluster. Using the approximate relation for main sequence stars that $L \propto M^4$, estimate the main sequence turnoff mass for each cluster. Using the approximate relation that main sequence lifetime scales as $t_{\rm MS} \propto M^{-3}$, estimate the age of each cluster. Compare with professional estimates.

   (c) Identify some of the obvious binary star systems in your HR diagram, and describe why they appear where they do.

4. **Realistic stellar structures with MESA** (10 points)

   (a) Create and evolve a $1M_\odot$ stellar model. Output a stellar profile when the age of your model is approximately 4.6 Gyr, the current age of the Sun.

   (b) Using your `profile*.data` file, make a plot of $\log\rho$, $\log P$, $\log T$ vs. radius for your model. By computing numerical derivatives, also plot $d\log T/d\log P$ as a function of radius. Indicate (e.g., by using a thick or colored line) which parts of the star are convective. Where and why is $d\log T/d\log P$ nearly constant?

   (c) Plot the nuclear energy generation rate, `eps_nuc`, as a function of radius. Make sure you edit your `profile_columns.list` file so that this quantity is included in the output. Compare `eps_nuc` to the approximate pp-energy generation rate $\epsilon = \epsilon_0 \rho T^4$, and estimate the proportionality constant ϵ_0.

   (d) Repeat part b for the red giant model you made in Homework 2. You will notice a dense core surrounded by a low-density envelope. Briefly describe the nature of the dense core and the low-density envelope.