Ay101 Problem Set 2

Due Monday, October 28

1. **Convection** (10 points)

   In class, we showed that the buoyant acceleration felt by a fluid element displaced radially by Δr is

$$\begin{aligned} a &= g\left[\frac{d\ln\rho}{dr} + \frac{g}{c_s^2}\right]\Delta r \\ &= -N^2\Delta r \end{aligned} \tag{1}$$

   where N is the Brunt-Vaisala frequency and $c_s^2 = \gamma P/\rho$. You may assume the unperturbed star is spherical and in hydrostatic equilibrium.

   (a) The Brunt-Vaisala frequency can also be expressed

$$N^2 = \frac{g}{H}\left(\nabla_{\rm ad} - \nabla\right),$$

   where $H = P/(\rho g)$ is the pressure scale height. The mixing length theory of convection envisions blobs of material that advect energy after traveling a "mixing length" Λ, parameterized in terms of the pressure scale height as $\Lambda = \alpha H$. If a convective blob accelerates from zero velocity over a mixing length according to

$$\frac{d}{dr}v_{\rm con} = |N|,$$

   find the maximum convective velocity $v_{\rm con}$ in terms of N and the mixing length Λ. You may assume N is constant over a mixing length. Express $v_{\rm con}$ in terms of α, γ, $(\nabla_{\rm ad} - \nabla)$, and c_s.

   (b) Using your expression for $v_{\rm con}$ from above, express the kinetic energy flux $F_{\rm con} = \rho v_{\rm con}^3$ of upgoing convective blobs.

   (c) The base of the Sun's convection zone has radius $r \sim 5\times10^{10}$ cm, density $\rho \sim 10^{-1}$ g/cm^3, sound speed $c_s \sim 2\times10^7$ cm/s, and scale height $H \sim 5\times10^9$ cm. For $\alpha = 2$, what is the value of $(\nabla_{\rm ad} - \nabla)$ required for convection to carry the Sun's luminosity? What is the corresponding maximum convective velocity $v_{\rm con}$, and how does this compare to the sound speed c_s?

   (d) Assuming convection carries all the Sun's luminosity, use the expression for N^2 in equation 1 to show that the density gradient in a convection zone is given by

$$\frac{d\ln\rho}{dr} = -\frac{g}{c_s^2}\left[\frac{\gamma^2}{\alpha^2}\left(\frac{L_\odot}{4\pi\rho r^2 c_s^3}\right)^{2/3} + 1\right] \tag{2}$$

   Using the Sun's properties from part c, determine which of the terms in the brackets in equation 2 is larger. Will the Sun's density profile be strongly dependent on α?

2. **Fully convective cool stars** (15 points)

   Cool stars have convective envelopes that contains most of the mass and volume of the star. Assume that the envelope is convective and composed of an ideal gas with polytropic index γ.

(a) In cool stars, the polytropic convective envelope extends nearly all the way to the photosphere. Use this fact to derive a scaling between pressure and temperature at the photoshpere.

(b) The photospheric pressure of a star is roughly

$$P_s = \frac{2g}{3\kappa_s}$$

where g is the gravitational acceleration and κ_s is the photospheric opacity. In stars cooler than the Sun, H- opacity is the dominant opacity near the photosphere, and has approximate dependence $\kappa_{H-} \propto T^9$. Use this fact, and your result from part a, to determine a scaling between $M, R, T_{\rm eff}$ for cool stars. Also express this scaling in terms of M, R, L, and $M, T_{\rm eff}, L$. **Hint:** You'll need to include the mass/radius dependence of the polytropic constant $K \propto M^{-1/2}R^{-3/2}$ for $n = 3/2$, see HKT Chapter 7.

(c) Main sequence G/K/M stars approximately have $L \propto M^3$. Derive a relation between L and $T_{\rm eff}$ for these stars, and draw the lower part of the main sequence on an HR dagram, from $0.1\,M_\odot < M < 1\,M_\odot$.

(d) As stars evolve up the red giant branch, they become nearly fully convective, their mass remains nearly constant, and their luminosity increases. Derive the scaling between luminosity and temperature for a red giant, and add an evolutionary track for a red giant branch star on an HR diagram.

3. **Field Color Magnitude Diagram** (15 points)

   Select the subset of all stars within 25 pc from *Gaia* with good colors (S/N in the blue and red bands $\geq$5) and excellent parallaxes (S/N $>$ 20).

   (a) Make a CMD ($BP - RP$ vs. M_G).

   (b) Make an H-R diagram using the effective temperature and luminosity given by *Gaia*. Remark on the similarities and differences between the CMD and H-R diagram.

   (c) Using the data for the effective temperature and $BP-RP$, find an approximation for temperature given a color, of the form $T = T_0 - T_1(BP - RP)$, and estimate reasonable values of T_0 and T_1.

   (d) Using the data for M_G, luminosity, and effective temperature, find a relationship for luminosity given M_G and $BP - RP$, of the form $L = L_\odot 10^{(M_g - M_0)/M_1}$, and estimate values of M_0 and M_1. What should these numbers be if M_g was a bolometric absolute magnitude?

   (e) Using your relations for luminosity and temperature given M_G and $BP-RP$, derive a relationship using these *Gaia* parameters to determine the radius of a star. Plot your inferred radius compared to the *Gaia* measured values for stars where you have a radius measurement, and comment on the difference.

4. **Evolution Tracks with MESA** (10 points)

   Evolve a $1.3\,M_\odot$ model up the red giant branch until its luminosity exceeds $100\,L_\odot$.

   (a) Edit the plotting bounds of the HR diagram in the `inlist_pgstar` file so that you can see the model evolve on an HR diagram.

   (b) Using the `history.data` located in the `LOGS` folder, make an HR diagram for your stellar evolution track. You may use whatever program you want (e.g., Python, IDL, etc.). `mesa_reader` is a useful too for analyzing MESA output, and can be downloaded at
   https://github.com/wmwolf/py_mesa_reader/
   with documentation at
   https://wmwolf.github.io/py_mesa_reader/

   (c) Plot the luminosity $\log L$ and central density ρ_c as a function of time. In each plot, label the main sequence, sub-giant, and red giant phases of evolution.