Ay101 Final Exam

Due Friday, December 13, 5:00 pm

Time allowed: 3 hours from when you open or print the PDF. Do not look beyond page 2 until you are ready to start the exam.

You may use one page (one side) of pre-written notes during the exam. Turn in this page of notes with the exam. Otherwise, you may not use the class notes, textbook, or online resource during the exam. You may use an unprogrammable calculator. Page 2 provides some possibly useful constants. You should have only a copy of this exam, your page of notes, blank paper, a pencil, and (optionally) a calculator.

Provide clear explanations for all of your equations. If we cannot follow your logic, then we cannot give you credit. For problems with multiple parts, you will sometimes need to use the answer you find in an earlier part of a problem to complete the rest of the question. If you cannot get an answer for the earlier part, take your best guess and use this answer to complete the rest of the problem. You will not be penalized multiple times for your previous incorrect answer.

Attempt only five questions of your choice.

Each question is worth 6 percentage points on your final grade. The entire exam is worth 30% of your final grade.

Please return solutions to Mia or Jim by email or in person, or under Jim's door.

This exam is to be taken under the guidelines outlined by Caltech's Honor Code.

Ay 123: Useful Constants and Units

Constant/Unit	Symbol	Value
Gravitational Constant	G	$6.67 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
Speed of light	c	$3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
Planck constant	h	$6.63 \times 10^{-27}\,\mathrm{erg\,s}$
	h	$4.14 \times 10^{-21}\,\mathrm{MeV\,s}$
	hc	$1240\,\mathrm{eV\,nm} = 1240\,\mathrm{MeV\,fm}$
	$\hbar c$	$197\,\mathrm{eV\,nm} = 197\,\mathrm{MeV\,fm}$
Avogadro number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
proton mass	m_p	$1.6726 \times 10^{-24}\,\mathrm{g}$
neutron mass	m_n	$1.6749 \times 10^{-24}\,\mathrm{g}$
α particle mass	m_α	$6.6442 \times 10^{-24}\,\mathrm{g}$
electron mass	m_e	$9.1 \times 10^{-28}\,\mathrm{g} = 0.511\,\mathrm{MeV}\,c^2$
electron charge	e	4.8×10^{-10} esu (cgs)
	e	1.6×10^{-19} C (SI)
	e^2	$1.44\,\mathrm{eV\,nm}$
Boltzmann constant	k_B	$1.38 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$
	k_B	$8.6 \times 10^{-5}\,\mathrm{eV\,K^{-1}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-5}\,\mathrm{erg\,cm^{-2}\,s^{-1}\,K^{-4}}$
Radiation constant	a	$4\sigma_{SB}c^{-1} = 7.5 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$
Rydberg constant	R	$2.18 \times 10^{-11}\,\mathrm{erg}$
	R	$13.6\,\mathrm{eV}$
Bohr radius	a_0	$5.3 \times 10^{-9}\,\mathrm{cm}$
Proton radius	r_p	$1 \times 10^{-13}\,\mathrm{cm}$
Thomson cross-section	σ_T	$6.7 \times 10^{-25}\,\mathrm{cm^2}$
Solar mass	$M_\odot$	$1.99 \times 10^{33}\,\mathrm{g}$
Solar radius	$R_\odot$	$7 \times 10^{10}\,\mathrm{cm}$
Solar bolometric luminosity	$L_\odot$	$4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$
Solar T_eff	$T_{\mathrm{eff},\odot}$	$5780\,\mathrm{K}$
Astronomical Unit	AU	$1.5 \times 10^{13}\,\mathrm{cm}$
parsec	pc	$3.26\,\mathrm{ly} = 3.1 \times 10^{18}\,\mathrm{cm}$
1 eV in erg		$1.6 \times 10^{-12}\,\mathrm{erg}$
1 Å		$10^{-8}\,\mathrm{cm}$
year		$\sim \pi \times 10^7\,\mathrm{s}$

Problem 1: Convection (8 points)

a) (4 points) By considering the stability of a parcel of gas that expands and contracts adiabatically, show using diagrams and text that the condition for stability in a star obeying the ideal gas law is

$$-\frac{dT}{dr} < \frac{\gamma - 1}{\gamma}\frac{\mu m_p g}{k_B}$$

Be sure to explain the physical assumptions.

b) (3 points) Use the relevant stellar structure equations for a radiative star to show that this equation reduces to

$$L(r) < \frac{\gamma - 1}{\gamma}\frac{16\pi acGm_p}{3k_B}\frac{T^3\mu M(r)}{\rho\kappa}$$

where $L(r)$ and $M(r)$ are the luminosity and enclosed mass at radius r, κ is the opacity, and a is the radiation constant.

c) (1 point) From the result above, give two reasons why convection tends to be triggered in regions of partial hydrogen/helium ionization within stars.

Problem 2: Star Clusters (8 points)

a) (2 points) Construct a Hertzsprung-Russel diagram of a newborn stellar cluster, assuming stars ranging from $0.1 - 100\,M_\odot$. Indicate the approximate maximum and minimum stellar luminosities (in units of $L_\odot$), assuming $L \propto M^4$. Indicate the approximate maximum and minimum stellar temperatures (in cgs units), assuming $T_{\text{eff}} \propto M^{1/2}$.

b) (2 points) The absolute magnitude of the Sun is roughly $M = 5$. Compute the absolute magnitude of the highest and lowest mass stars in the cluster.

c) (2 points) Estimate how the main-sequence lifetime of a star scales with its mass. Compute the main-sequence turnoff mass for clusters of age 10^7 and 10^{10} years. Construct HR diagrams of clusters with these ages.

d) (2 points) In your cluster of age 10^{10} years, indicate the approximate locations of the red giant branch and the white dwarf cooling track. Are there any white dwarfs in the younger cluster, and why or why not?

Problem 3: Timescales, Temperature, and Diffusion (8 points)

a) (1 point) Explain the Kelvin-Helmholtz timescale. What is the Kelvin-Helmholtz timescale for the Sun?

b) (1 point) Provide an estimate for the lifetime of the Sun on the main sequence.

c) (3 points) Estimate the central temperature of the Sun from first principles.

d) (3 points) Discuss why radiative transport is a diffusion process. For a typical opacity of $\kappa = 1\,\text{cm}^2/\text{g}$, calculate the typical mean free path of a photon in the Sun, and estimate the time it takes for a photon from the core of the Sun to reach the surface.

Problem 4: Stellar Structure Equations (8 points)

For increasing mass along the main sequence, the dominant source of opacity changes, the dominant nuclear reaction changes, and the dominant source of pressure changes. We'll assume that the change in nuclear energy generation occurs at 1 $M_\odot$, the change in opacity occurs at 5 $M_\odot$, and the change in equation of state happens at 25 $M_\odot$. You therefore have four mass regimes: (1) $M < M_\odot$, (2) $M_\odot \leq M \leq 5\,M_\odot$, (3) $5\,M_\odot \leq M \leq 25\,M_\odot$, and (4) $M \geq 25\,M_\odot$.

a) (2 points) Write down the four equations of stellar structure, assuming radiative energy transport. In a sentence, describe the meaning of each equation.

b) (1 point) Provide a fifth equation needed to solve the stellar structure equations, in both regimes 1-3 and regime 4, and briefly state its meaning.

c) (3 points) For each regime, describe

   (a) How and why the nuclear energy generation rate scales with central temperature.

   (b) How and why the opacity scales with internal density and temperature.

   (c) How and why the central pressure scales with density and temperature.

d) (1 point) Use the central temperature of the Sun and its average density to estimate the ratio of radiation to gas pressure in its interior.

e) (1 point) Identify which of the five structure equations will change for a white dwarf, and provide at least one new equation applicable to these stars.

Problem 5: Polytropes (8 points)

A polytropic equation of state is one that satisfies

$$P = K\rho^{1+1/n}$$

where K and n are independent of radius.

a) (2 points) Give two situations in stellar evolution for which polytropic solutions are appropriate, explaining clearly the value of n that is applicable in each case.

b) (2 points) What value of n corresponds to an object of uniform density? Make plots of density as a function of radius for a polytrope with a large value of n, compared to a polytrope with a small value of n.

c) (2 points) For a polytropic equation of state, the pressure and internal energy density per unit mass, E, are related by $P = \rho E/n$. The total energy is simply $W = U + \Omega$, where U is the total internal energy of the star. Using the Virial theorem, derive an equation that relates W and Ω.

d) (2 points) A stable star must have negative total energy. Use this fact to derive a stability criterion (in terms of n) based on your result above.

Problem 6: White Dwarfs (8 points)

a) (3 points) Starting from the expression for pressure

$$P = \frac{1}{3}\int_0^\infty v\,p\,\frac{dn}{dp}dp\,, \tag{1}$$

and using the Pauli exclusion principle

$$\frac{dn}{dp} \leq \frac{8\pi p^2}{h^3}, \tag{2}$$

Show that the electron pressure in a completely degenerate plasma is

$$P = \frac{8\pi c}{3h^3} \int_0^{p_F} \frac{p/(m_e c)}{\sqrt{1 + p^2/(m_e c)^2}} p^3 \, dp \tag{3}$$

where m_e is the electron rest mass, and p_F is the Fermi momentum. Express the Fermi momentum in terms of the electron number density n_e.

b) (3 points) Consider how this expression can be simplified in the non-relativistic and ultra-relativistic limits. By equating relativistic and non-relativistic degeneracy pressure, derive the critical density, ρ_{crit}, for the transition between these two extremes assuming, for simplicity, that $n_e = \rho/m_{\text{H}}$ in determining the electron number density.

c) (2 points) In simple terms (without further derivations), explain why the radius of a degenerate, non-relativistic white dwarf becomes smaller as its mass increases and why there is a maximum stable mass in the ultra-relativistic case.

Problem 7: Nuclear Reactions (8 points)

a) (1 point) Calculate the mean thermal energy of a proton in the core of a star where hydrogen is burning at a temperature of 10^7 K.

b) (2 points) Estimate the closest distance such energy could bring two protons and compare this with the proton radius.

c) (2 points) Using the fact that in a Maxwellian distribution, the number of particles above energy E falls off exponentially, estimate the fraction protons in the core of the Sun with enough energy to overcome the Coulomb barrier.

d) (3 points) Without any lengthy derivation, explain how the Coulomb barrier is overcome and why, for non-resonant reactions, there is a preferred energy at which fusion occurs.

Problem 8: Helioseismology (8 points)

a) (3 points) Use the linearized momentum equation

$$\rho \frac{\partial}{\partial t} v = -\frac{\partial}{\partial x} \delta P \tag{4}$$

the linearized continuity equation

$$\frac{\partial}{\partial t} \delta\rho = -\rho \frac{\partial}{\partial x} v \, . \tag{5}$$

and adiabatic equation of state

$$\frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho} \tag{6}$$

Assuming constant density ρ and pressure P, derive a wave equation and show that pressure disturbances travel at the sound speed

$$c_s^2 = \frac{\gamma P}{\rho} \, .$$

b) (3 points) From your wave equation, show that the spacing between a star's oscillation mode frequencies is $\Delta\omega = \pi c_s/R$. Show (e.g., using the Virial theorem) that this implies $\Delta\omega \sim \sqrt{GM/R^3}$, where M and R are the mass and radius of the star.

c) (2 points) Summarize what can be measured from quantitative studies of pressure waves within stars like the Sun.

Problem 9: Stellar Evolution (8 points)

a) (1 point each) For each of the following phases of a 1 $M_\odot$ star's evolution, describe the basic stellar structure, the approximate surface properties (R and $T_{\rm eff}$), and the dominant internal power source.

   (a) Hayashi track
   (b) Main sequence
   (c) Red giant branch
   (d) Red clump
   (e) Asymptotic giant branch
   (f) White dwarf

b) (2 points) Which of these phases of evolution is avoided in a 3 $M_\odot$ star and why?